How does gravity work? An answer to that question might be found by noting that gravitational mass and inertial mass are equal. That may mean that gravitational and inertial forces are related. The study of inertial forces is mainly the study of momentum and its relationship to force. Consider a general case of an impressed force on a mass m. Relative to a stationary reference frame B, the force impressed on mass m results in a change in its vector momentum $\overrightarrow{P_B}$ according to:

$$\overrightarrow{P_B} = \left(\frac{mc\overrightarrow{\beta_B}}{\sqrt{1-\beta_B^2}} \right)$$

$$\overrightarrow{P_B} = P_{XB}\,\overrightarrow{i} + P_{YB}\,\overrightarrow{j} \qquad \overrightarrow{\beta_B} = \beta_{XB}\,\overrightarrow{i} + \beta_{YB}\,\overrightarrow{j}$$

$\overrightarrow{i}, \overrightarrow{j}$ - unit vectors in the x, y directions

$F_{XB} = dP_{XB}/dt_B$ - force on mass m in the x-direction

$F_{YB} = dP_{YB}/dt_B$ - force on mass m in the y-direction

$$F_{XB} = \left(\frac{mc}{(1-\beta_{XB}^2-\beta_{YB}^2)^{3/2}} \right) \{(1-\beta_{YB}^2)A_{XB} + \beta_{XB}\beta_{YB}A_{YB}\} \qquad (20a)$$

$$F_{YB} = \left(\frac{mc}{(1-\beta_{XB}^2-\beta_{YB}^2)^{3/2}} \right) \{(1-\beta_{XB}^2)A_{YB} + \beta_{XB}\beta_{YB}A_{XB}\} \qquad (20b)$$

$A_{XB} = \left(\frac{d\beta_{XB}}{dt_B} \right)$ - (x-direction acceleration of mass m)/c

$A_{YB} = \left(\frac{d\beta_{YB}}{dt_B} \right)$ - (y-direction acceleration of mass m)/c

Equation (20) can be applied to experiments where force and momentum are involved. For example, consider the experiment of Figure 7. A stationary reference frame B is pushing on a mass m with forces F_{XB} and F_{YB}. Mass m is stationary in a reference frame A that is moving with x-direction velocity β_X (the velocity in the y-direction $\beta_Y = 0$).

This seems to be a simple case at first glance, however the experiment of Figure 7 lacks critical information. The forces F_{XB} and F_{YB} are stationary in frame B but are applied to a mass moving in the x-direction. The analysis of any experiment involving forces will require the mechanisms of the force application to be well defined. Force F_{XB} could be applied by an observer or a spring pressing on mass m in the direction of its velocity. But force F_{YB} requires mechanism that can push perpendicular to the velocity. One way to apply this force to moving mass m is shown in Figure 8.

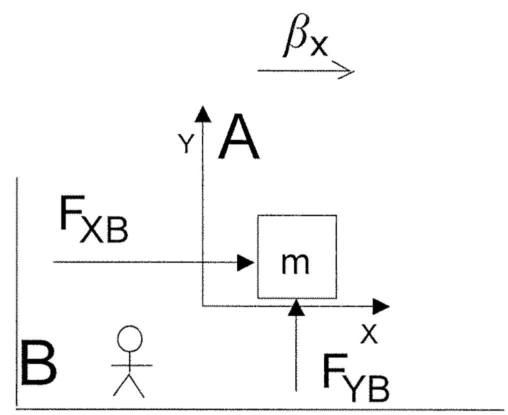

Figure 7

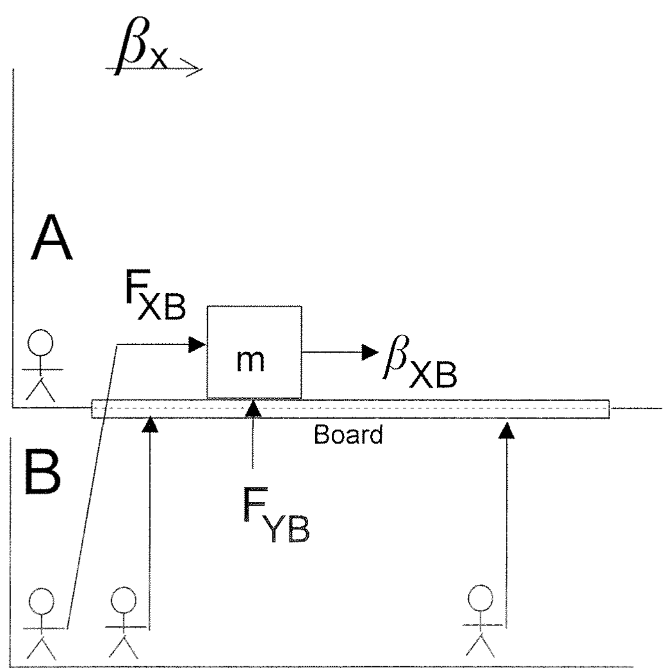

Figure 8

In Figure 8, the left frame B observer is providing the x-direction force on mass m. Two other frame B observers are pushing vertically on a board. Mass m is moving with velocity β_{XB} relative to frame B. The board is frictionless and pushing against mass m. Frame A is moving with velocity $\beta_X = \beta_{XB}$. The observers are careful to keep the board parallel to velocity β_X, regardless of the situation of mass m. They are also careful to insure that the total force they impress on the board is a constant $F_{YB} = F_b$. At the instant shown in Figure 8, the mass m has no y-direction velocity. No momentum change of frame B occurs, as the experiment is actually run twice. A mirror image experiment with exactly opposite forces is run simultaneously by frame B, but is not shown.

It is desired to find the relationship between the forces applied by frame B and the forces that mass m "feels", which would be those forces measured by a reference frame in which mass m is stationary. This happens to be frame A. Applying (20):

$$F_{XB} = \left(\frac{mcA_{XB}}{(1-\beta_{XB}^2)^{3/2}} \right) \tag{21a}$$

$$F_{YB} = \left(\frac{mcA_{YB}}{(1-\beta_{XB}^2)^{1/2}} \right) \tag{21b}$$

Applying the conditions of this experiment to (16) and (17):

$$A_{XB} = A_{XA}(1-\beta_{XB}^2)^{3/2} \tag{22a}$$

$$A_{YB} = A_{YA}(1-\beta_{XB}^2) \tag{22b}$$

The forces that mass m "feels" are the forces F_{XA} and F_{YA} observed in reference frame A. Noting that $F_{XA} = mcA_{XA}$ and $F_{YA} = mcA_{YA}$, the final result is:

$$F_{XB} = F_{XA} \tag{23a}$$

$$F_{YB} = F_{YA}\sqrt{1-\beta_{XB}^2} \tag{23b}$$

The result (23) gives transformations for forces simultaneously applied between inertial reference frames. These equations are similar to other well known transformations in Special Relativity. However, (23) is only true for the exact instant where there is no y-direction velocity to mass m.

The case where there is a y-direction velocity to mass m is worthy of investigation. In Figure 9, the experiment is exactly the same as the experiment of Figure 8 except that mass m is moving with velocity β_{TB}, with components β_{XB} and β_{YB}. Also note that the force F_{XB} has now been omitted from the experiment.

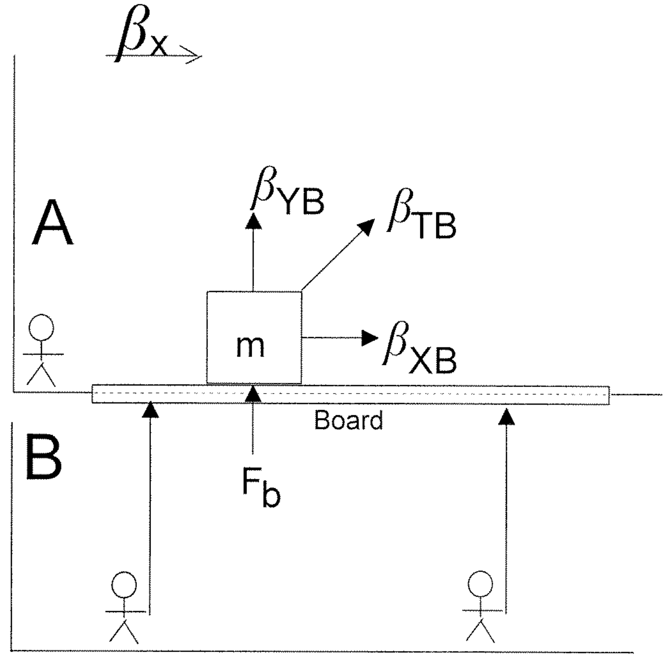

Figure 9

Frame B sees mass m move in response to any impressed force as described by (20). In this case, F_{XB} is zero, so (20a) gives:

$$A_{XB} = -\left(\frac{\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2}\right) \tag{24}$$

The frame B observers see an acceleration of mass m in the x-direction, even though the experiment is specifically structured to avoid x-direction force application. More discussion on this result follows later. They also see mass m move vertically in response to their impressed vertical force F_b. Inserting (24) into (20b) gives the reaction of mass m as frame B sees it.

$$F_b = \left(\frac{mcA_{YB}}{\sqrt{1-\beta_{XB}^2-\beta_{YB}^2}\,(1-\beta_{YB}^2)}\right) \tag{25}$$

As before, in order to find the forces that mass m "feels", it is necessary to define a frame M in which mass m is stationary. Frame M is shown in Figure 10. Frame M is moving upward at velocity β_{YA} relative to frame A.

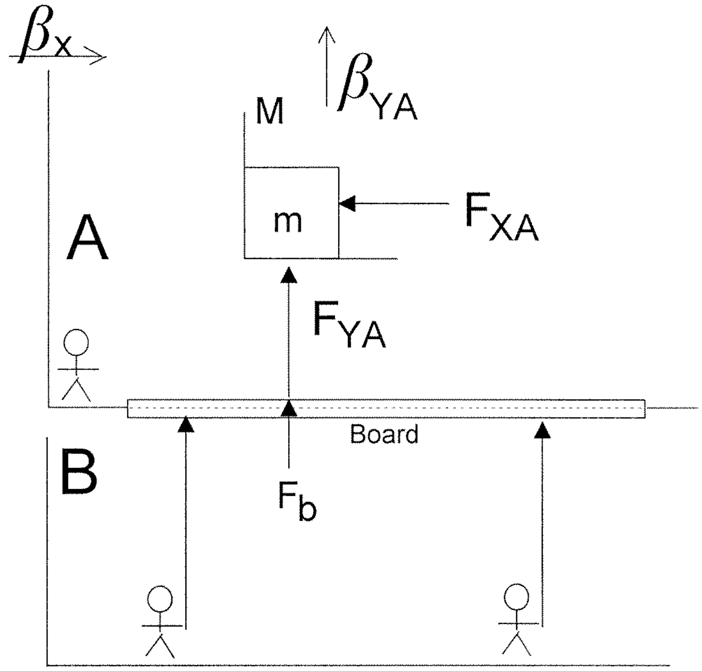

Figure 10

In Figure 10, the frame A velocity is $\beta_X = \beta_{XB}$, so that $\beta_{XA} = 0$ for mass m. Also, $\beta_Y = 0$ for frame A. Mass m is stationary in frame M at the instant shown. It is now known from (24) that frame A sees a force F_{XA} applied to mass m. How the board applies this force will be explained later. For now, the relationship of the forces in Figure 10 can be written down by adapting (20) to the frame A point of view.

$$F_{YA} = \left(\frac{mcA_{YA}}{(1-\beta_{YA}^2)^{3/2}} \right) \tag{26a}$$

$$F_{XA} = \left(\frac{mcA_{XA}}{(1-\beta_{YA}^2)^{1/2}} \right) \tag{26b}$$

The relationship between β_{YA} and β_{YB} can be found from (13).

$$\beta_{YA} = \left(\frac{\beta_{YB}}{\sqrt{1-\beta_{XB}^2}} \right) \tag{27}$$

The relationship between A_{XA} and A_{XB} can be found from (16).

$$A_{XB} = A_{XA}(1 - \beta_{XB}^2)^{3/2} \tag{28}$$

The relationship between A_{YA} and A_{YB} can be found from (17) and (24).

$$A_{YA} = A_{YB}\left(\frac{1 - \beta_{XB}^2 - \beta_{YB}^2}{(1 - \beta_{XB}^2)^2(1 - \beta_{YB}^2)}\right) \tag{29}$$

Inserting (27) and (28) into (26b):

$$F_{XA} = \left(\frac{mcA_{XB}}{\sqrt{1 - \beta_{XB}^2 - \beta_{YB}^2}\,(1 - \beta_{XB}^2)}\right) \tag{30}$$

Inserting (27) and (29) into (26a):

$$F_{YA} = \left(\frac{mcA_{YB}}{\sqrt{1 - \beta_{XB}^2 - \beta_{YB}^2}\,(1 - \beta_{YB}^2)\sqrt{1 - \beta_{XB}^2}}\right) \tag{31}$$

And:

$$\left(\frac{F_{XA}}{F_{YA}}\right) = \left(\frac{-\beta_{XB}\beta_{YB}}{\sqrt{1 - \beta_{XB}^2}}\right) \tag{32}$$

In this example, there is a horizontal force as seen by frame A, even though frame B only applies a vertical force. So, where does F_{XA} come from? The answer to that question is that frame B is pushing on mass m in the x-direction even though the experiment has been designed to specifically avoid this. The way this occurs is shown by viewing the experiment from the instantaneous point of view of frame A, as is shown in Figure 11.

Frame A sees frame B going by with velocity $-\beta_{XB}$. The distance between the two observers impressing the vertical force on the board is L and frame A sees this distance as $L\sqrt{1 - \beta_{XB}^2}$. The board has velocity β_{YB} as seen by frame B. The observer in frame B that is on the right in Figure 11 is 'behind' the other observer, so his clock is showing a later time by $L\beta_{XB}/c$. In this time period, the board travels distance $L\beta_{YB}\beta_{XB}$ as seen by either frame. So the right observers section of the board has gone further that the left observers section. Frame A sees the board tilted and the force applied (that is perpendicular) to the board is tilted. Therefore, simple geometry gives:

$$\left(\frac{F_{XA}}{F_{YA}}\right) = \left(\frac{-L\beta_{XB}\beta_{YB}}{L\sqrt{1 - \beta_{XB}^2}}\right) = \left(\frac{-\beta_{XB}\beta_{YB}}{\sqrt{1 - \beta_{XB}^2}}\right) \tag{33}$$

This is the same result as equation (32). The alteration of the space-time geometry of the experiment as viewed by frame A is responsible for the appearance of the x-direction force on mass m. This is the method of the application of the forces to mass m leading to (26). Because mass m is not actually stationary in frame A, this is only an approximate solution. It is reasonably accurate for low values of β_{XB} and β_{YB}, but loses accuracy as those velocities approach light speed.

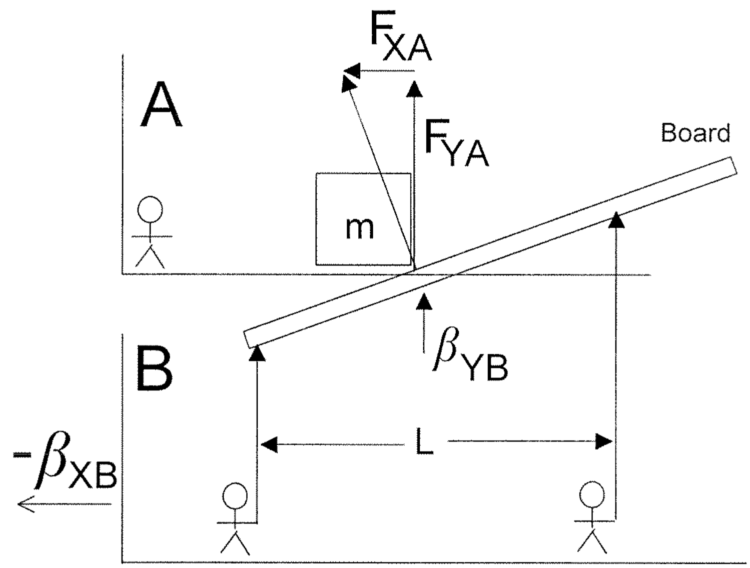

Figure 11

The above explanation gives a simple visual statement of the source of the forces that mass m feels. For an exact derivation of these forces, it is necessary to define frame A as one in which mass m is stationary and then to use equations (16) and (17) to describe the actual accelerations. If mass m is stationary in frame A, then:

β_{XB}, β_{YB} - x and y direction velocities of mass m relative to frame B

β_X, β_Y - x and y direction velocities of frame A relative to frame B

$$\beta_X = \beta_{XB} \qquad \beta_Y = \beta_{YB} \qquad \beta_T^2 = \beta_X^2 + \beta_Y^2$$

$$F_{XA} = mcA_{XA} \qquad F_{YA} = mcA_{YA} \qquad (34)$$

Equation (24) can be inserted into factor K_B used in (16) and (17) to give:

$$K_B = A_{YB}\left(\frac{\beta_Y - \left(\frac{\beta_X\beta_{XB}\beta_{YB}}{1-\beta_{YB}^2}\right)}{1-\beta_X\beta_{XB}-\beta_Y\beta_{YB}}\right) \tag{35}$$

And, using the definition of K_{TB} from (12) and (13):

$$S_1 = 1-\left(\frac{\beta_X^2}{\beta_T^2}\right)(1-\sqrt{1-\beta_T^2})$$

$$S_2 = 1-\left(\frac{\beta_X\beta_Y}{\beta_T^2}\right)(1-\sqrt{1-\beta_T^2})$$

$$S_3 = 1-\left(\frac{\beta_Y^2}{\beta_T^2}\right)(1-\sqrt{1-\beta_T^2})$$

$$A_{XA} = ((\left(\frac{-\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2}\right)+\beta_{XB}K_B)S_1-(A_{YB}+\beta_{YB}K_B)S_2)K_{TB}^2 \tag{36a}$$

$$A_{YA} = ((A_{YB}+\beta_{YB}K_B)S_3-(\left(\frac{-\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2}\right)+\beta_{XB}K_B)S_2)K_{TB}^2 \tag{36b}$$

$$\left(\frac{F_{XA}}{F_{YA}}\right)=\left(\frac{A_{XA}}{A_{YA}}\right) \tag{37}$$

To see the relationship of the board to mass m under these circumstances, consider Figure 12.

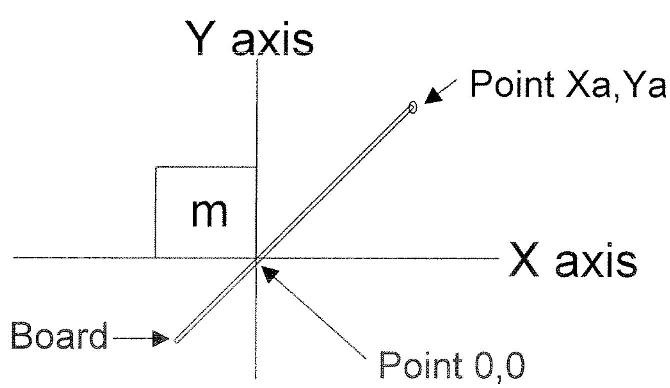

Figure 12

In Figure 12, frame A and frame B are instantaneously on top of each other with the origins coinciding at point 0,0. The board is oriented to give mass m a force in the x and y directions, much as it did in Figure 11. In this position, a point Xa, Ya on one end of the board describes the board's orientation. Equations (7) and (11) can be used to define the points shown at the instant $t_A = 0$ when mass m is instantaneously stationary in frame A. Under these conditions, $\beta_{XA} = 0$ and $\beta_{YA} = 0$. Point 0,0 is where the board touches mass m at the origins of both frames. Using (7), it is clear that point 0,0 is located at the following coordinates:

$$x_A = 0 \qquad y_A = 0 \qquad x_B = 0 \qquad y_B = 0 \tag{38}$$

Using (11) with $t_A = 0$, frame A sees the clock readings of coordinate points in frame B as:

$$t_B = \left(\frac{x_B \beta_X}{c}\right) + \left(\frac{y_B \beta_Y}{c}\right) \tag{39}$$

From the point of view of frame B, the following conditions apply at any coordinate point x_B, y_B along the board:

$$x_B = \text{constant}$$

$$y_B = c\beta_{YB} t_B \tag{40}$$

Combining (40) with (39) will show how frame A sees the clock readings and board coordinates along the board.

$$t_B = \left(\frac{x_B \beta_X}{c(1 - \beta_Y \beta_{YB})}\right)$$

$$y_B = \left(\frac{x_B \beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}}\right) \tag{41}$$

Now, (7) can be used to give:

$$x_A = x_B(S_1 - \left(\frac{\beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}}\right) S_2) \tag{42a}$$

$$y_A = x_B(\left(\frac{\beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}}\right) S_3 - S_2) \tag{42b}$$

Based upon simple geometry such as illustrated in Figure 11:

$$\left(\frac{F_{XA}}{F_{YA}}\right) = -\left(\frac{y_A}{x_A}\right) \tag{43}$$

Equations (36), (37) and (42) do not look as though they would reduce to (43). However, if these equations are put into a computer program to give numerical values to each of the quantities, it will be found that (43) is exactly true for all values of inputs. The geometric configuration of the board gives the exact ratio of forces to produce the movement of mass m as seen by frame B.

By setting $\beta_X = \beta_{XB}$ and $\beta_Y = \beta_{YB}$, (37) and (43) give results for the forces that mass m "feels" in this experiment. Because the equations leading up to (37) and (43) still have separate values for β_X, β_Y and β_{XB}, β_{YB}, these equations can also be used to determine the forces and board relationships for other reference frames moving with general velocities (mass m is not stationary in these reference frames). For example, if $\beta_Y = 0$ is used as an input, (37) and (43) will verify the accuracy of (33) and (32) at lower values of mass m velocity.

The major point of this example is that a vertical force F_b applied by frame B results in a horizontal force on mass m. The horizontal force that appears on mass m in this example is, in many ways, like a gravitational force in the horizontal direction on mass m.

To further illustrate this point, consider Figure 13. An observer A is standing on the surface of the planet. Another observer B is falling in the gravitational field of the planet. The observer A feels the planet push on his feet in a direction from right to left in the Figure. Observer B also sees observer A accelerate from right to left. In the experiment of Figure 9, the mass m 'feels' a push in the horizontal direction from right to left. Reference frame B also sees mass m accelerate in a direction from right to left (equation (24)).

The observer A in Figure 13 sees observer B accelerating from left to right, but observer B does not feel any acceleration. In the experiment of Figure 9, mass m sees Frame B accelerating from left to right, but the Frame B observers report that they do not feel any horizontal force.

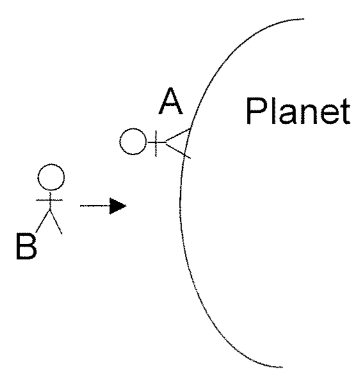

Figure 13

To summarize, the situation of falling in a gravitational field is similar to an inertial reference frame experiment where nothing is happening. In other words, just being an inertial reference frame means there is no effect noticed by observers in that frame as they move in constant motion. But this inertial frame can interact with (apply forces to) other frames in a direction perpendicular to the motion of the reference frame without feeling an acceleration in the direction of motion.

Also, accelerating in a gravitational field (such as standing 'motionless' on the surface of a gravitational mass) is similar to an inertial experiment where an object is receiving an applied force in a direction perpendicular to the direction of the gravity-like force. The inertial mass of the experiment of Figure 9 experiences the horizontal gravity-like force in that experiment similar to what an observer standing on a planet would feel. Both of these forces could be called Mysterious Forces, as they have no explicit source as normally defined by the action-reaction logic of traditional inertial Newtonian physics.

The General Theory of Relativity describes a space-time geometry distortion in a gravitational field surrounding a mass, resulting in a gravitational acceleration of objects in that field. What is unique about the experiment of Figure 9 is that it shows visually how a distorted space-time geometry can produce a Mysterious Force where inertial Newtonian physics and 'normal common sense' would indicate that no force should exist. The dynamic (inertial) calculations (the Law of Conservation of Momentum) also agree that this Mysterious Force exists. This gravity-like force is a special case of pure inertial physics in Special Relativity.