The Ladder Paradox is familiar to most people who have studied Special Relativity. It's about a man running with a ladder into a garage. When at rest, the ladder is longer than the garage. But at speed, the ladder fits into the garage. Although there is a comical quality to the story, it is useful for explaining length contraction and "failure of simultaneity at a distance." Almost as an afterthought, it is mentioned at the end of the story that a friend shuts the door to the garage once the ladder is completely inside. The man stops running and the ladder crashes through the walls as it reverts to its original length.

This experiment is in danger of breaking the Law of Conservation of Energy. Consider the experiment from the start. The man is stationary outside the garage. He starts running toward the garage and expends energy E to get up to speed. When he gets into the garage, he slows back down and gets all of this initial energy E back again. Shutting the door takes zero energy. When the ladder crashes through the garage after becoming stationary, isn't the energy generated by this crash created by the experiment? The answer to that question is that energy is not created and the reason why gives some fundamental insight into the process of acceleration.

Figure 1 shows a version of this experiment. Two objects are moving with the velocities v_1 and v_2 and travel toward each other. The length of the bar when stationary is L_1 and is greater than L_2, but the length of the bar at velocity v_1 is $L_1(1-(v_1/c)^2)^{1/2}$, which is less than L_2. By coincidence, the two objects come together as illustrated in the instant shown in Figure 2.

The two objects have missed hitting each other and the bar is just inside the space between the ends of the U-shaped object. In this moment, the velocity of both objects is brought to zero. In

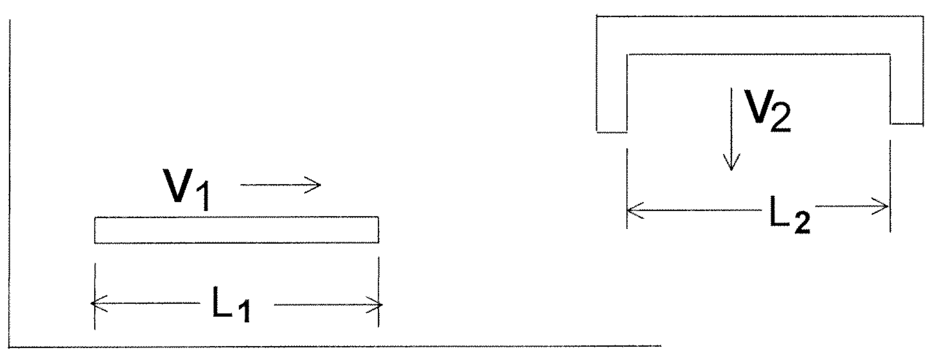

Figure 1

accounting for the energy of this experiment, bringing the velocity of the objects to zero yields kinetic energy recovered from the objects which would be exactly equal to the energy used to accelerate the objects to their respective speeds initially. However, the bar is now trapped inside

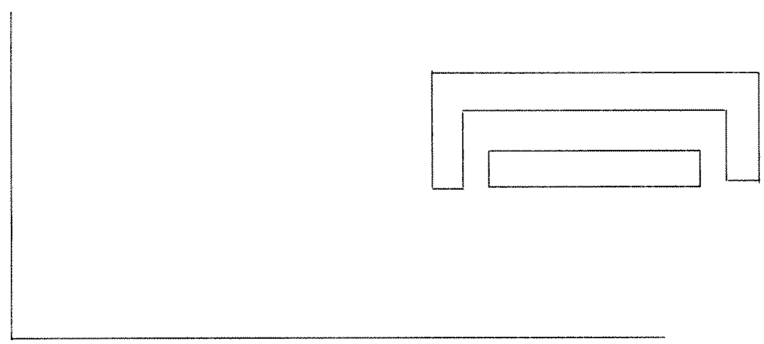

Figure 2

the U-shaped object. The length of the bar while stationary is greater than the distance L_2 so the bar is now confined to a space that is too small for it, as in the original ladder paradox. There would be a stress energy occurring in the materials of the two objects caused by this confinement.

To resolve this paradox, the process of acceleration needs examination. But first, some simple terminology will be defined. Refer to Figure 3, where reference frame A is traveling with velocity v relative to reference frame B. Observers in frame A measure the length of the bar (stationary in frame A) to be L_A and frame B observers measure its length to be L_B.

$$L_B = L_A(1 - (v/c)^2)^{1/2} \qquad (1)$$

v = velocity of moving frame c = velocity of light

The observers in frame B measure the length of this bar to be shorter than L_A as described by (1). If a clock is located in frame A and advances at rate T_A and a clock in frame B advances at a rate T_B, then:

$$T_A = T_B(1-(v/c)^2)^{1/2} \qquad (2)$$

The clock in frame A, as seen by frame B, appears to run slow. Clocks can be placed on either end of this bar. As the bar is stationary in frame A, these clocks are synchronized as seen from frame A. From frame B the clock at the front of the bar would be reading earlier than the clock at the rear of the bar by Δt.

$$\Delta t = -L_A v/c^2 \qquad (3)$$

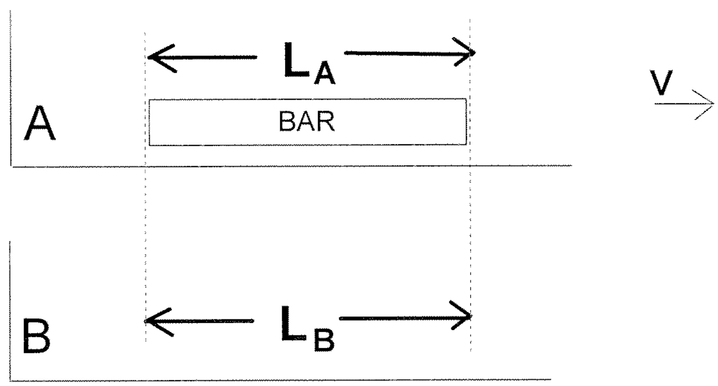

Figure 3

Now, the process of changing velocity will be examined. Figure 4 shows a circular object in a simple experiment where it changes velocity. The object starts the experiment at the origin of reference frame B where it is stationary. It then accelerates to velocity v where it is stationary in reference frame A. The acceleration process takes the object over a distance D, as measured by frame B.

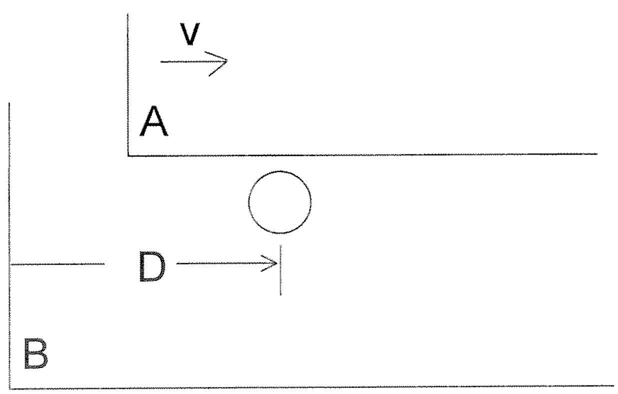

Figure 4

Now, a more complex experiment is shown in Figure 5. This experiment is the same as Figure 4 except that two objects (labeled 1 and 2) are initially stationary in frame B and simultaneously experience identical accelerations to velocity v. They both end up stationary in frame A after the acceleration ends. Both objects require the same distance D to get to velocity v (as observed by frame B). Object 1 starts at the frame B origin and object 2 starts at coordinate L_B. In frame B, the objects are distance L_B apart at the start of the experiment. Frame B also sees both objects a distance L_B apart when they are stationary in frame A at the end of the experiment. But since the moving frame A has its coordinate axis affected by length contraction, observers in frame A would see the objects a distance L_A apart.

$$L_A = \left(\frac{L_B}{(1-(v/c)^2)^{1/2}} \right) \qquad (4)$$

After the acceleration, frame A observers measure the two objects to be farther apart than they were when measured by observers in frame B before the acceleration.

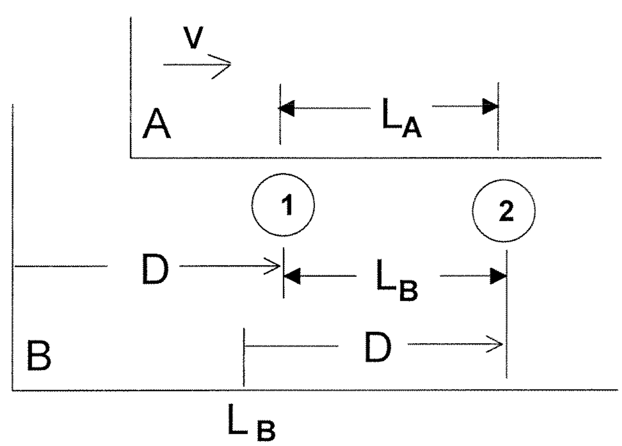

Figure 5

A similar experiment could be done at this point which would take the objects from frame A back to being stationary in frame B. The objects are stationary in frame A and accelerate identically from frame A (as seen by frame A). When the objects become stationary in frame B, the distance between them (as seen by frame B) will be L_{BB}:

$$L_{BB} = \left(\frac{L_A}{(1-(v/c)^2)^{1/2}} \right) = \left(\frac{L_B}{1-(v/c)^2} \right) \qquad (5)$$

Every time this type of acceleration occurs, the objects get farther apart. The reason the objects spread apart during each portion of the cycle is partly related to length contraction (equation (1)) and partly related to "failure of simultaneity at a distance" (equation (3)). This is explained by considering what happens in Figure 5 from the point of view of frame A.

Frame A sees frame B approaching at velocity -v with object 1 in the lead and object 2 trailing behind. The distance between the objects is L_B as measure by frame B. Frame A sees object 2 start to slow down first (because its clock is later) and object 1 starts to slow down $L_B v/c^2$ later (frame B time). Frame A would see each object experience the same motion as it accelerates (just as frame B sees identical motions for each object). If one object requires distance D to complete its acceleration, then so must the other. As seen by frame A, the time interval between the points when the objects start accelerating is $\left(\frac{L_B v}{c^2}\right)\left(\frac{1}{\sqrt{1-(v/c)^2}}\right)$. In this time interval, frame A sees frame B move a distance $\left(\frac{L_B v^2}{c^2}\right)\left(\frac{1}{\sqrt{1-(v/c)^2}}\right)$. Therefore, the total distance between the objects at the points where they start accelerating is L_A:

L_A= Initial distance apart + distance moved between object accelerations

$$L_A = L_B(1-(v/c)^2)^{1/2} + \left(\frac{L_B v^2}{c^2}\right)\left(\frac{1}{\sqrt{1-(v/c)^2}}\right)$$

$$L_A = \left(\frac{L_B}{\sqrt{1-(v/c)^2}}\right) \qquad (6)$$

So, after the objects are stationary in frame A, frame A sees the distance between the objects as (6), (4) or (1).

Now the experiment will be changed slightly, as shown in Figure 6. The experiment is the same as the one of Figure 5 but the two objects are now connected by a spring. The spring is unstressed when the objects are stationary in frame B at the start of the experiment. Each object undergoes the same motion as in Figure 5. However, in Figure 6, when the objects get to velocity v and are stationary in frame A, the spring will be stretched to accommodate the longer distance between the objects as described by (6). This stretching requires energy and this extra spring energy must be supplied by the mechanism doing the acceleration.

The way this occurs is as follows. As object 2 is accelerated, the spring tends to pull it backward, so additional force must be applied to object 2 to maintain the prescribed acceleration. As object 1 is accelerated, the spring tends to pull it forward, so less force is required for this object to maintain the desired acceleration. The total energy expended is given by integrating the forces on the objects (F_1 and F_2) over distance D. This equals the sum of the object's kinetic energies (KE_1 and KE_2) and the energy SE stored in the spring.

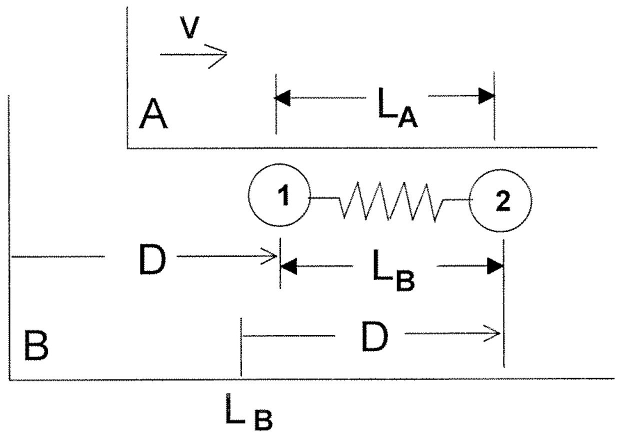

Figure 6

$$\int F_1 dD + \int F_2 dD = KE_1 + KE_2 + SE \qquad (7)$$

If the spring rate is zero, then the objects accelerate independently. No energy is stored in the spring and the experiment is the same as Figure 5. If the spring rate is not zero, then the experiment ends with less clarity. The spring starts with no tension when at rest in frame B. As the acceleration process ends, the spring will have a tension imposed by forces F_1 and F_2. The moment that these forces go to zero, the objects will then be pulled together by the spring and a vibration will start between them.

The location of the ends of this "flexible rod" would be indeterminant as long as the vibration continues. It is probable that the motions of the objects will eventually be damped after the experiment and a stable system with an unstressed spring will result. The vibrational energy will be converted to heat that will eventually be transferred to the environment.

In a variation of this experiment, frame B observes two objects with a spring between them traveling by at velocity v. See Figure 7. At the start of the experiment, when the objects are stationary in frame A, the spring is unstressed and the objects are a distance L_A apart as measured by frame A. Frame B observes the objects to be a distance L_B apart. This is shown in the top view of Figure 7 labeled "Deceleration Starts." Frame B observes the two objects simultaneously undergo identical decelerations to a velocity of zero. When the objects reach zero velocity, frame B will still see the objects a distance L_B apart. But since L_A is longer than L_B, the spring will now be compressed. This is shown in the bottom view of Figure 7 labeled "Deceleration Completed."

The spring becomes extended or compressed during an acceleration based upon which frame starts with the stationary objects and which frame sees the identical movement of the objects.

The discussion on stress energy in this experiment is similar to the discussion of Figure 6. The extra energy of the compressed spring must be supplied by the mechanism doing the deceleration. If frame B does the deceleration, the kinetic energy received by the deceleration mechanism is reduced by the amount of stress energy stored in the spring.

$$\int F_1 dD + \int F_2 dD = -KE_1 - KE_2 + SE \quad (8)$$

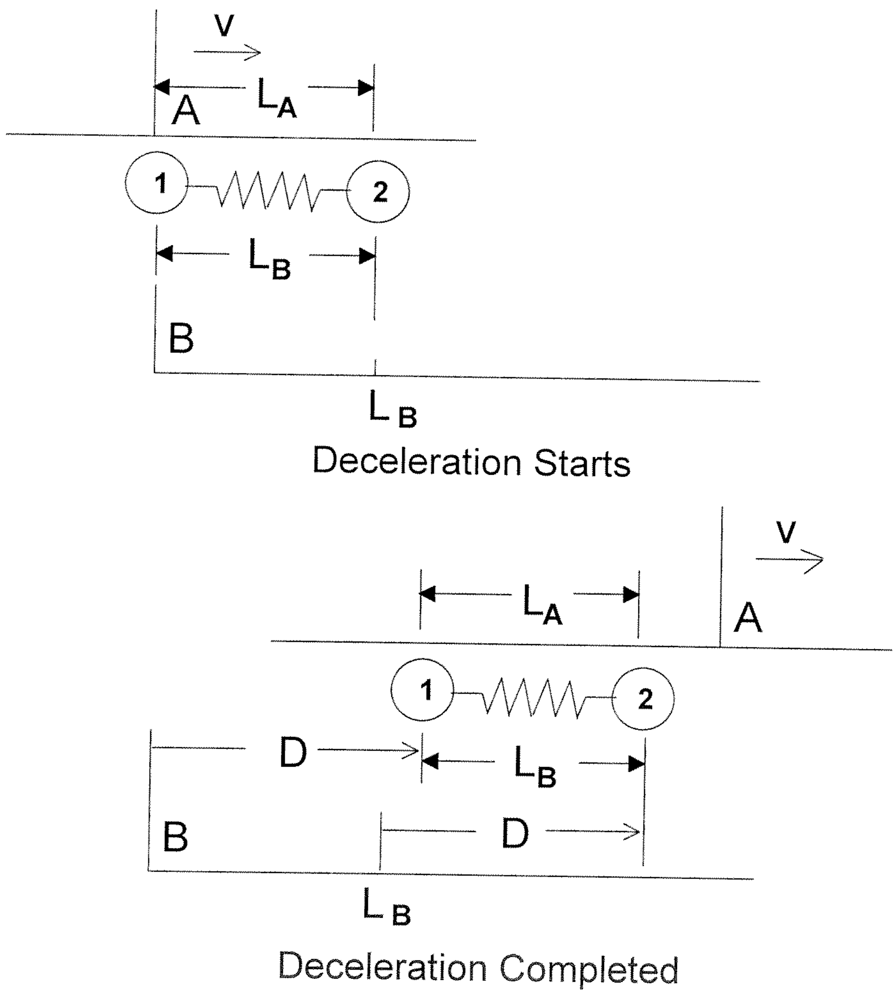

Figure 7

The minus sign in front of the KE term means there is a negative energy expended by the deceleration mechanism during the deceleration. From (8), the maximum possible spring energy is limited to an amount equal to the sum of the kinetic energies of the objects. If it were greater, then the deceleration mechanism would not receive energy during the deceleration, but would have to supply energy to decelerate the objects.

The experiments of Figures 6 and 7 could be a simple model for the material in our real universe. Material rods are composed of atoms held together by a collection of different forces. Some of these forces pull the atoms together and some push them apart. The combination of these effects results in an equilibrium position of the atoms within the material that maintains the dimensional stability of the material over time. These forces behave like the springs shown above which allow the material to be compressed or stretched. When real materials vibrate, these vibrations are eventually damped out by a variety of mechanisms. This vibratory energy is eventually turned into heat.

Now the ladder paradox can be analyzed. In the experiment of Figures 1 and 2, the bar would have to be accelerated to velocity v_1 as the first part of the experiment. This acceleration would stretch the bar. This would produce stress energy within the bar. This stress energy was supplied by the accelerating mechanism in addition to the energy that was supplied to accelerate the bar to speed v_1. After the acceleration ends, there would be a vibration of the material making up the bar. The vibration energy would be eventually dissipated and the length of the bar would become stable in length according to (1).

But this initial stress energy is not the energy trapped in the bar as shown in Figure 2 . That energy comes when the bar is decelerated back to zero velocity relative to frame B. Again, as the material of the bar decelerates, energy must be supplied to compress the "springs" between the atoms of the material. This stress energy is supplied by the deceleration mechanism. The amount of energy recovered by the deceleration mechanism is the kinetic energy gained minus the compressive stress energy supplied. The stress energy in the parts as shown in Figure 2 is taken from the kinetic energy recovered during the deceleration. The total energy available before and after the deceleration is the same. The Law of Conservation of Energy is satisfied.

In summary, this article presents one possible method to account for the energy of the ladder as it crashes through the walls of the garage. In actual real world experiments, the above condition (where the ends of the accelerating object are held to a strict original length throughout the acceleration) is probably not a realistic expectation of what will actually happen. But it would show a 'tendency'. Because real world materials are flexible, the magnitude of the acceleration to relativistic velocities of real world materials will also have other limitations.