

In the article *The Acceleration Law*, Special Relativity was used to show how various quantities on accelerating objects varied with distance L along the object. There is additional information of interest that is not given in that analysis. To fill in this missing information, consider the rocket shown in Figure 11.

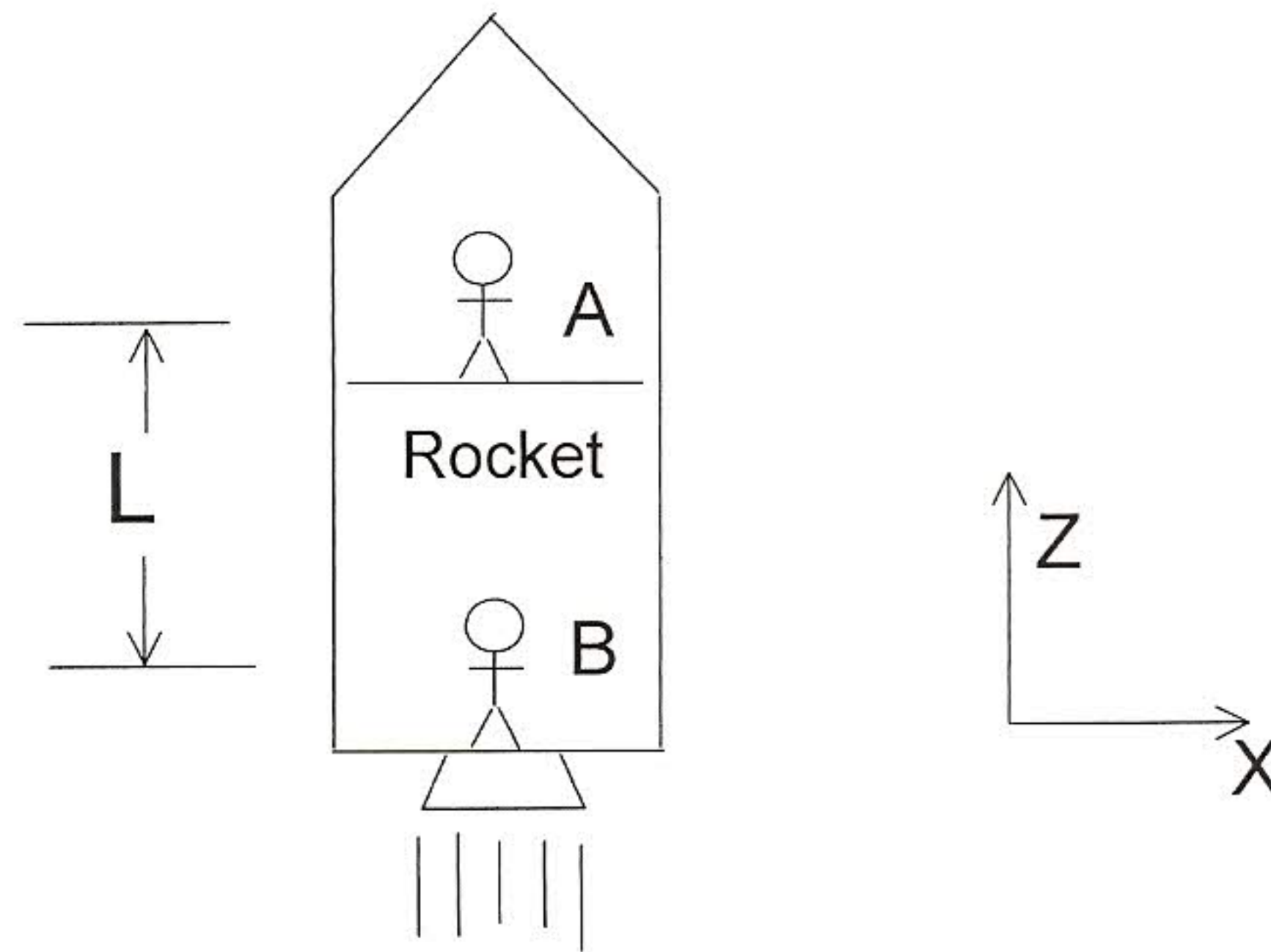


Figure 11

Observer B has acceleration a_B and observer A will have acceleration a_A as defined by (21). A comparison of length dimensions in the “horizontal” direction within the rocket is shown in Figure 12. If, during the acceleration, the observer in reference frame A “drops” a mass down to observer B, it would be expected that the mass would travel straight down in the z direction, not moving to the left or right during the fall. Similarly, two masses dropped from observer A to observer B would both be expected to fall straight (neglecting their small gravitational attraction for each other). But how can the observers be sure that the distance between the masses is the same in frame A at the beginning of the fall and when they arrive in frame B?

The way this is accomplished is to place a spring between the two masses. This spring could be thought of as a measuring device in its undeflected state (the spring has the same undeflected length dx in both reference frames). As the two masses move from frame A to frame B, the amount of energy recovered would be FL for each mass (F is the “weight” of each mass caused by the rocket acceleration and L is the vertical distance between the two reference frames). The energy obtained from the movement of the two masses must follow the Law of Conservation of Energy. If the spring were deflected after the movement, this deflection would result in additional spring potential energy that would not be consistent with the Law of Conservation of Energy. This result can be summarized in the transformation:

$$dx_A = dx_B \quad (40)$$

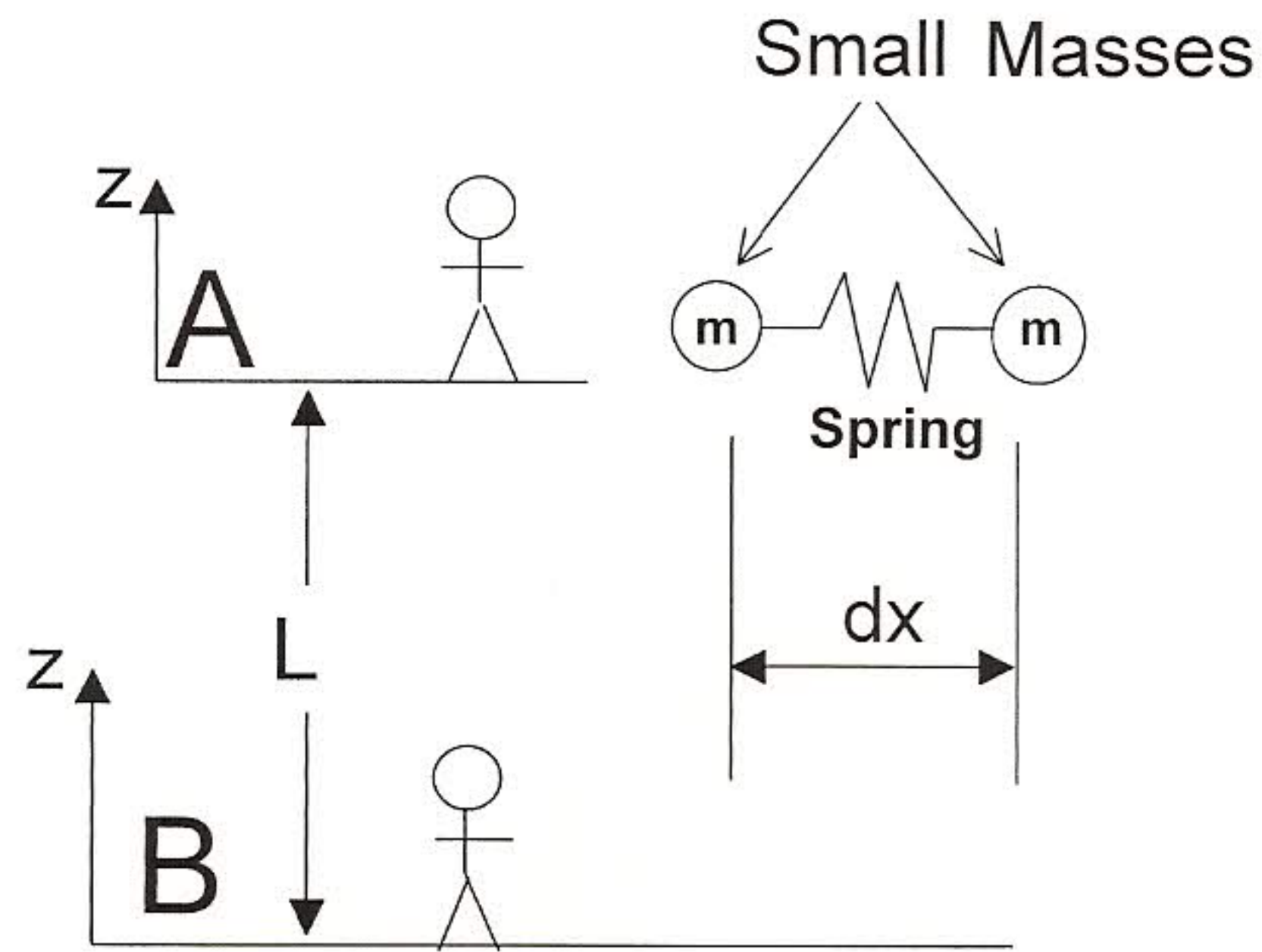


Figure 12

For an experiment showing the relationship of forces applied perpendicular to the acceleration direction, see Figure 13.

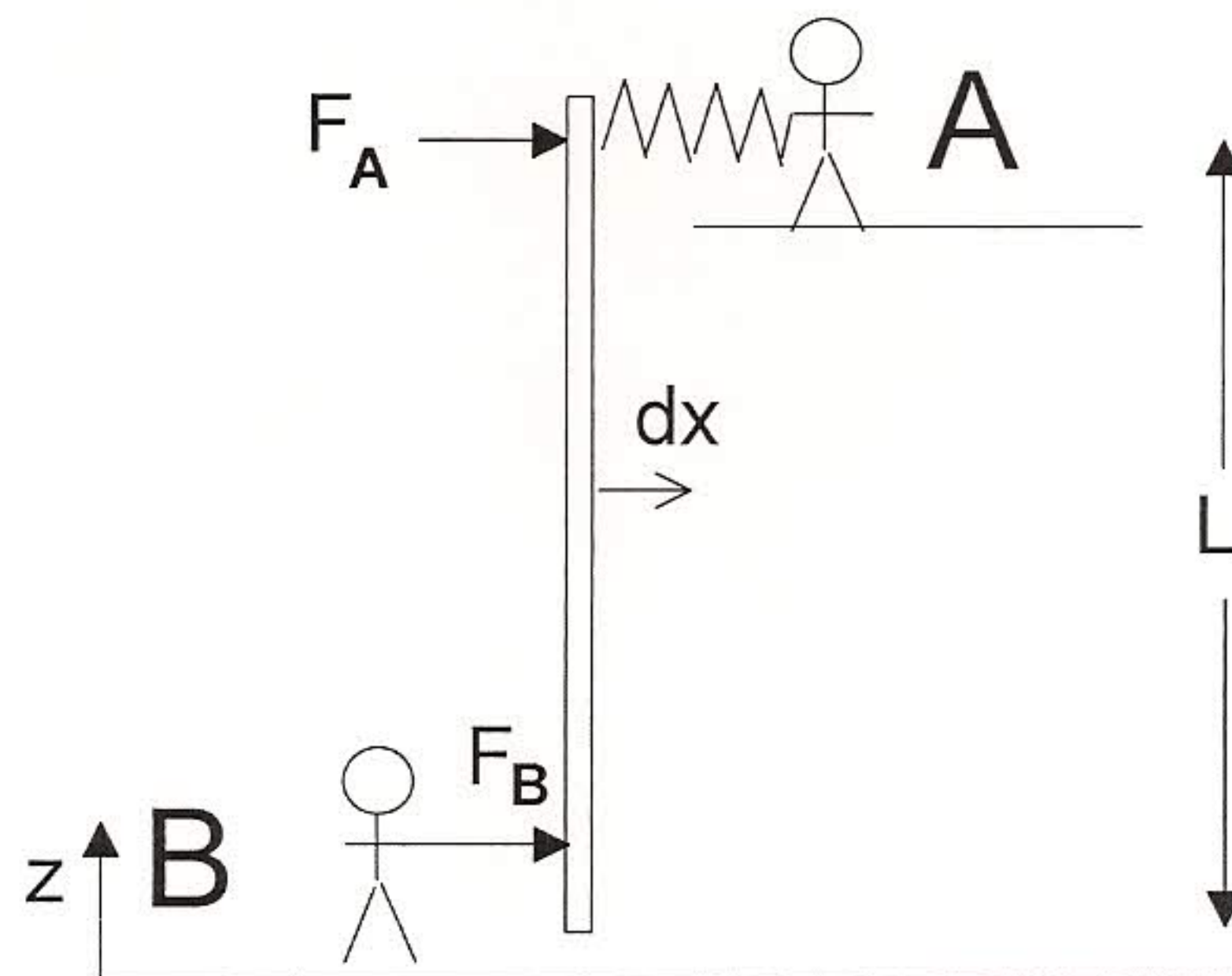


Figure 13

In Figure 13, observer B pushes on a long board that is constrained to only move horizontally (not rotate). Observer B also pushes on another board in a mirror image experiment (not shown) where the force is opposite to F_B . This eliminates any problems with angular or horizontal momentum in the experiment. Force F_B is applied by observer B but force F_A is felt at frame A. The spring in frame A is compressed, clamped in the compressed position and relocated to frame B, similar to the thought experiment concerning vertical forces in *The Acceleration Law*. So, (34) also gives the transformation for horizontal forces at different vertical positions (L coordinates).

Inertial observers watching the rocket see the time transformation (31) or (33). But what are the practical realities of the differing times at the two levels within the rocket? See Figure 14.

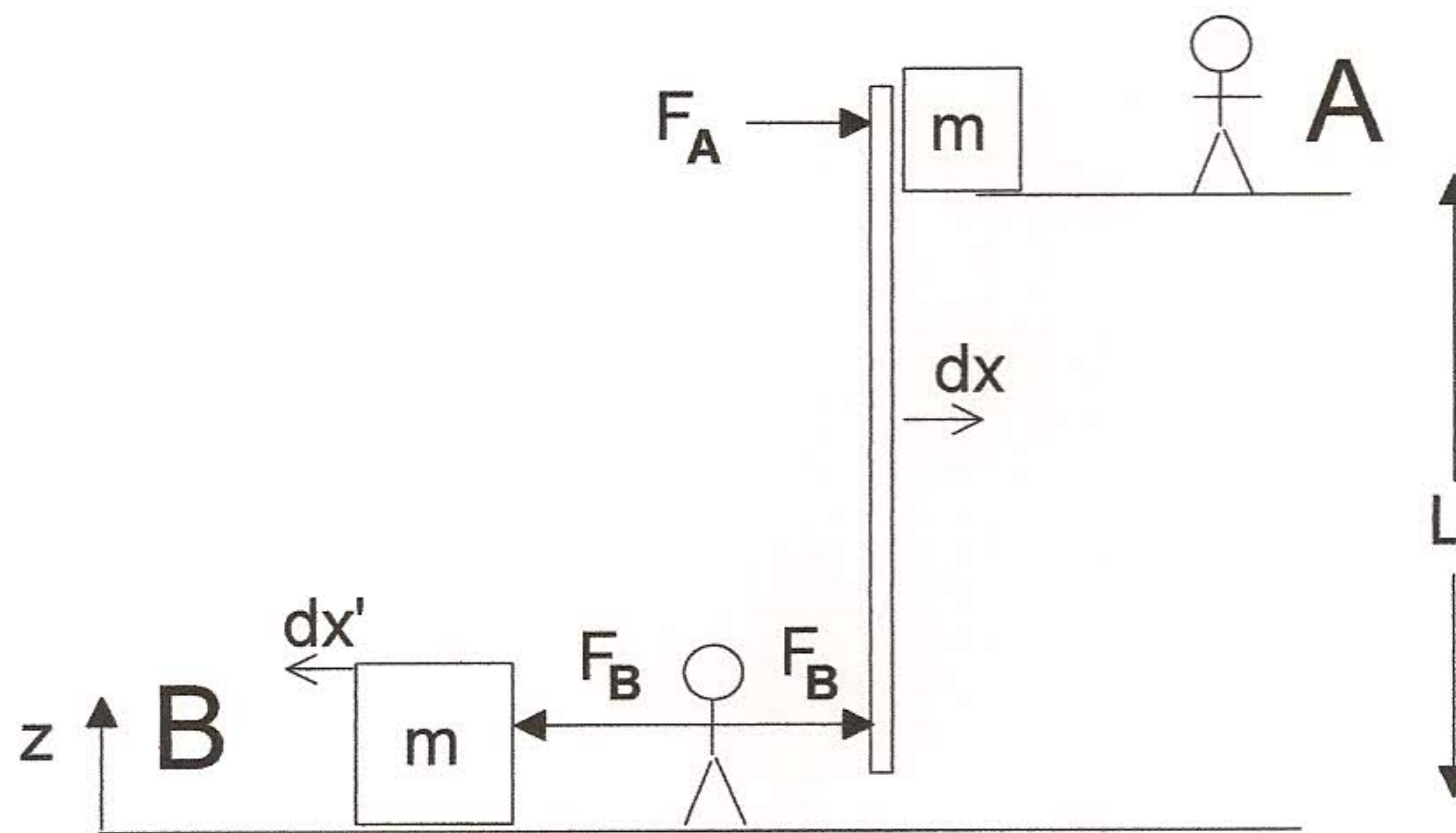


Figure 14

In Figure 14, a board is constrained to move only in the horizontal direction as in the experiment of Figure 13. However, this time the frame B observer applies a force F_B to the board and simultaneously to a second mass m located in frame B. The board pushes the mass in frame A to the right. The mass in frame B will travel to the left. The experiment starts with both masses stationary. Observer B applies the forces for a time period dt_B and the observer in frame A sees the force at his location applied for time period dt_A . The momentum produced in frame B must be equal in magnitude (and opposite in direction) to the momentum produced in frame A.

The change in momentum of either mass is given by the force applied and time period of force application.

$$F_A dt_A = F_B dt_B$$

$$\left(\frac{dt_A}{dt_B}\right) = 1 + \left(\frac{a_B L}{c}\right) \quad (41)$$

Equation (41) gives the relative time rates in the two reference frames as viewed by frame B.

In the experiment of Figure 14, observer B is supplying the energy and force to accelerate the two masses. For a time period dt_B , he simultaneously applies a force F_B to mass m in frame B and to the board. The board displacement as measured by frame B is dx_B . The board displacement as measured by frame A is dx_A . The mass in frame B is accelerated over a distance dx'_B . From (40), the board displacement is the same when measured in frame A and frame B ($dx_A = dx_B$). After a period of time t_B , the mass and board will have traveled distances x' and x respectively.

$$x'_B = \left(\frac{mc^2}{F_B}\right) \left[\left(1 + \left(\frac{F_B t_B}{mc}\right)^2\right)^{1/2} - 1 \right] \quad (42a)$$

$$x_A = \left(\frac{mc^2}{F_A}\right) \left[\left(1 + \left(\frac{F_A t_A}{mc}\right)^2\right)^{1/2} - 1 \right] \quad (42b)$$

Assume all displacements are positive in the directions shown in Figure 14. In the next incremental time periods dt_B and dt_A , the two masses will travel incremental distances dx'_B and dx_A . The result is:

$$dx'_B = \left(\frac{\left(\frac{F_B t_B}{m}\right) dt_B}{\left(1 + \left(\frac{F_B t_B}{mc}\right)^2\right)^{1/2}} \right)$$

$$dx_A = \left(\frac{\left(\frac{F_A t_A}{m}\right) dt_A}{\left(1 + \left(\frac{F_A t_A}{mc}\right)^2\right)^{1/2}} \right)$$

$$F_A t_A = F_B t_B$$

$$dx_A = dx_B = dx'_B \left(1 + \left(\frac{a_B L}{c}\right)\right) = dx'_A \left(1 + \left(\frac{a_B L}{c}\right)\right) \quad (43)$$

The board moves farther than the frame B mass under the same force F_B . The frame A observer cannot measure the distance dx'_B directly like the frame B observer. What he can do is touch the mass in frame B with another board and follow its movement without applying a force to accelerate it. This movement measured by frame A is dx'_A and $dx'_A = dx'_B$. This result is also shown in (43).

Defining $\beta_B = \left(\frac{dx_B}{cdt_B}\right)$, $\beta'_B = \left(\frac{dx'_B}{cdt_B}\right)$, $\beta_A = \left(\frac{dx_A}{cdt_A}\right)$ and $\beta'_A = \left(\frac{dx'_A}{cdt_A}\right)$:

$$\beta'_A \left(1 + \left(\frac{\alpha_B L}{c}\right)\right) = \beta'_B = \beta_A = \left(\frac{\beta_B}{1 + \left(\frac{\alpha_B L}{c}\right)}\right) \quad (44)$$

At this point, the horizontal (x-direction) momentum will be calculated as a check of the assumption that the mass m momentum in frame A is equal to the mass m momentum in frame B. To do this, both masses must be brought into the same reference frame so that their momentums can be compared directly. The frame A mass will be placed in a box and “dropped” down to frame B. See Figure 15. At the instant that the box was “dropped” from frame A, F_B was made zero, so that mass m in frame B still has the same velocity β'_B that it had when mass m in frame A had velocity β_A . The box actually doesn't drop. It is traveling at a constant velocity (inertial reference frame) and frame B is accelerating up to it. At the instant shown in Figure 15, the two masses are at the same vertical coordinate within the rocket ship. The box has no x-direction velocity, so mass m is still traveling with velocity β_A within it. Meanwhile, frame B has accelerated to velocity β^* . The momentum of mass m in the box is:

$$P_A = \left(\frac{mc\beta_A}{\sqrt{1 - \beta_A^2}}\right) \quad (45)$$

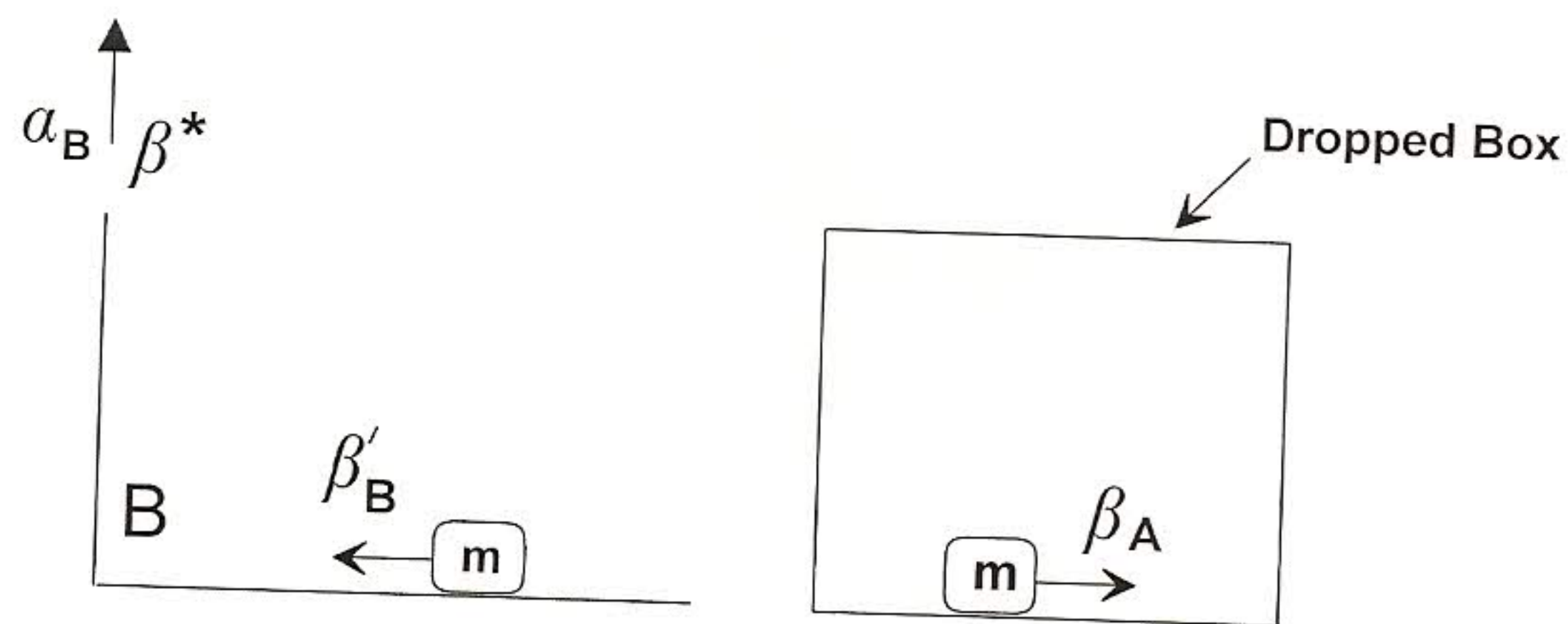


Figure 15

The velocity of mass m in frame B must now be calculated relative to the box. This will be done using (14) from the article *2 Dim. Position, Velocity, Acceleration* from the series of articles on Force and Space-Time. The reference frame labels in that article are backwards from the labels used in Figure 15, so care must be taken to keep track of the terms in this calculation. Quantities of Figure 15 will be shown in square parentheses and quantities of (14) will be shown without.

$$\begin{aligned}
 \beta_X &= [0] & \beta_Y &= [\beta^*] & \beta_T &= [\beta^*] & \text{Accelerating reference frame B velocities} \\
 \beta_{XA} &= [-\beta'_B] & \beta_{YA} &= [0] & & & \text{Velocities of mass in accelerating reference frame} \\
 K_S &= 1 - \sqrt{1 - [\beta^*]^2} & & & K_{TA} &= \sqrt{1 - [\beta^*]^2} \\
 \beta_{XB} &= [\beta'_B] \sqrt{1 - [\beta^*]^2} & & & \beta_{YB} &= [\beta^*] & (46)
 \end{aligned}$$

The velocity components of the accelerating frame B mass relative to the inertial box reference frame are β_{XB} and β_{YB} as shown in (46). The x-direction momentum of the frame B mass relative to the inertial box reference frame is P_{XB} and :

$$\begin{aligned}
 P_{XB} &= \left(\frac{mc\beta_{XB}}{\sqrt{1 - \beta_{XB}^2 - \beta_{YB}^2}} \right) = \left(\frac{mc[\beta'_B] \sqrt{1 - [\beta^*]^2}}{\sqrt{1 - [\beta'_B]^2} \sqrt{1 - [\beta^*]^2}} \right) \\
 P_{XB} &= P_A & (47)
 \end{aligned}$$

Mass also undergoes a relativistic transformation at different vertical positions within the rocket. Because $dx_B \neq dx'_B$, the observer in frame B measures the mass in frame A to be different than m . He believes it is another value which he calls m_{AB} (the mass in frame A as measured from frame B). His deduction is:

$$\begin{aligned}
 dx_B &= \left(\frac{\left(\frac{F_{Bt_B}}{m_{AB}} \right) dt_B}{\left(1 + \left(\frac{F_{Bt_B}}{m_{AB}c} \right)^2 \right)^{1/2}} \right) & dx'_B &= \left(\frac{\left(\frac{F_{Bt_B}}{m} \right) dt_B}{\left(1 + \left(\frac{F_{Bt_B}}{mc} \right)^2 \right)^{1/2}} \right) \\
 m_{AB} &= m \left(\frac{\beta'_B}{\beta_B} \right) \sqrt{\frac{1 - \beta_B^2}{1 - (\beta'_B)^2}} & (48)
 \end{aligned}$$

To the frame B observer, it “feels” as if m_{AB} is less than m , as defined by (48). The observer in frame A knows the frame B observer applies the same force to the frame B mass and board. This force is F_A when measured by the frame A observer. The frame A observer does not know the value of the mass in frame B, so he calls it m_{BA} (the mass in frame B as determined by frame A). He would then conclude:

$$dx_A = \left(\frac{\left(\frac{F_{At_A}}{m} \right) dt_A}{\left(1 + \left(\frac{F_{At_A}}{mc} \right)^2 \right)^{1/2}} \right) \quad dx'_A = \left(\frac{\left(\frac{F_{At_A}}{m_{BA}} \right) dt_A}{\left(1 + \left(\frac{F_{At_A}}{m_{BAC}} \right)^2 \right)^{1/2}} \right)$$

$$m_{BA} = m \left(\frac{\beta_A}{\beta'_A} \right) \sqrt{\frac{1 - (\beta'_A)^2}{1 - \beta_A^2}} \quad (49)$$

The frame A observer would conclude that the mass in frame B is larger than m , as defined by (49).

Equation (47) showed that the actual horizontal momentums of the masses were equal when one mass is brought into the others reference frame, but what does observer B think is the momentum of the mass in frame A? He calls this momentum P_{AB} and the momentum of the mass in frame B is P'_B . Using (44) and (48):

$$P_{AB} = \left(\frac{m_{ABC} \beta_B}{\sqrt{1 - \beta_B^2}} \right) = \left(\frac{mc \beta'_B}{\sqrt{1 - \beta_B'^2}} \right) = P'_B \quad (50)$$

The frame B observer is satisfied that the momentum's of the two masses are equal and that the Law of Conservation of Momentum has been observed.

The frame A observer will calculate the momentum of frame B mass relative to frame A as P'_{BA} :

$$P'_{BA} = \left(\frac{m_{BAC} \beta'_A}{\sqrt{1 - (\beta'_A)^2}} \right) = \left(\frac{mc \beta_A}{\sqrt{1 - \beta_A^2}} \right) = P_A \quad (51)$$

The frame A observer is also satisfied that the Law of Conservation of Momentum has been observed.

The frame B observer applies force F_B to the frame B mass over a distance dx'_B and to the board over a distance dx_B . This means that he does not send an equal amount of energy to the mass and board. If the energy sent to the frame B mass is W'_B and the energy sent to the board is W_B , then:

$$W_B = F_B dx_B$$

$$W'_B = F_B dx'_B = F_B \left(\frac{dx_B}{1 + \left(\frac{a_B L}{c} \right)} \right) = \left(\frac{W_B}{1 + \left(\frac{a_B L}{c} \right)} \right) \quad (52)$$

The work that is transferred to the mass in frame A is W_A .

$$W_A = F_A dx_A = \left(\frac{F_B}{1 + \left(\frac{a_B L}{c} \right)} \right) dx_B = \left(\frac{W_B}{1 + \left(\frac{a_B L}{c} \right)} \right) \quad (53)$$

The work delivered to the two masses in the two different reference frames is the same. The kinetic energy of the frame B mass is KE' .

$$KE' = \left(\frac{mc^2}{\sqrt{1 - (\beta'_B)^2}} \right) - mc^2 \quad (54)$$

The kinetic energy of the frame A mass is KE .

$$KE = \left(\frac{mc^2}{\sqrt{1 - \beta_A^2}} \right) - mc^2 = \left(\frac{mc^2}{\sqrt{1 - (\beta'_B)^2}} \right) - mc^2 = KE' \quad (55)$$

The kinetic energies of the two masses are equal. This concludes the discussion of Figure 14, where the forces are applied in a direction perpendicular to the direction of the rocket acceleration. For the case where the forces on the m masses are applied in a direction parallel to the direction of the rocket acceleration, see Figure 16.

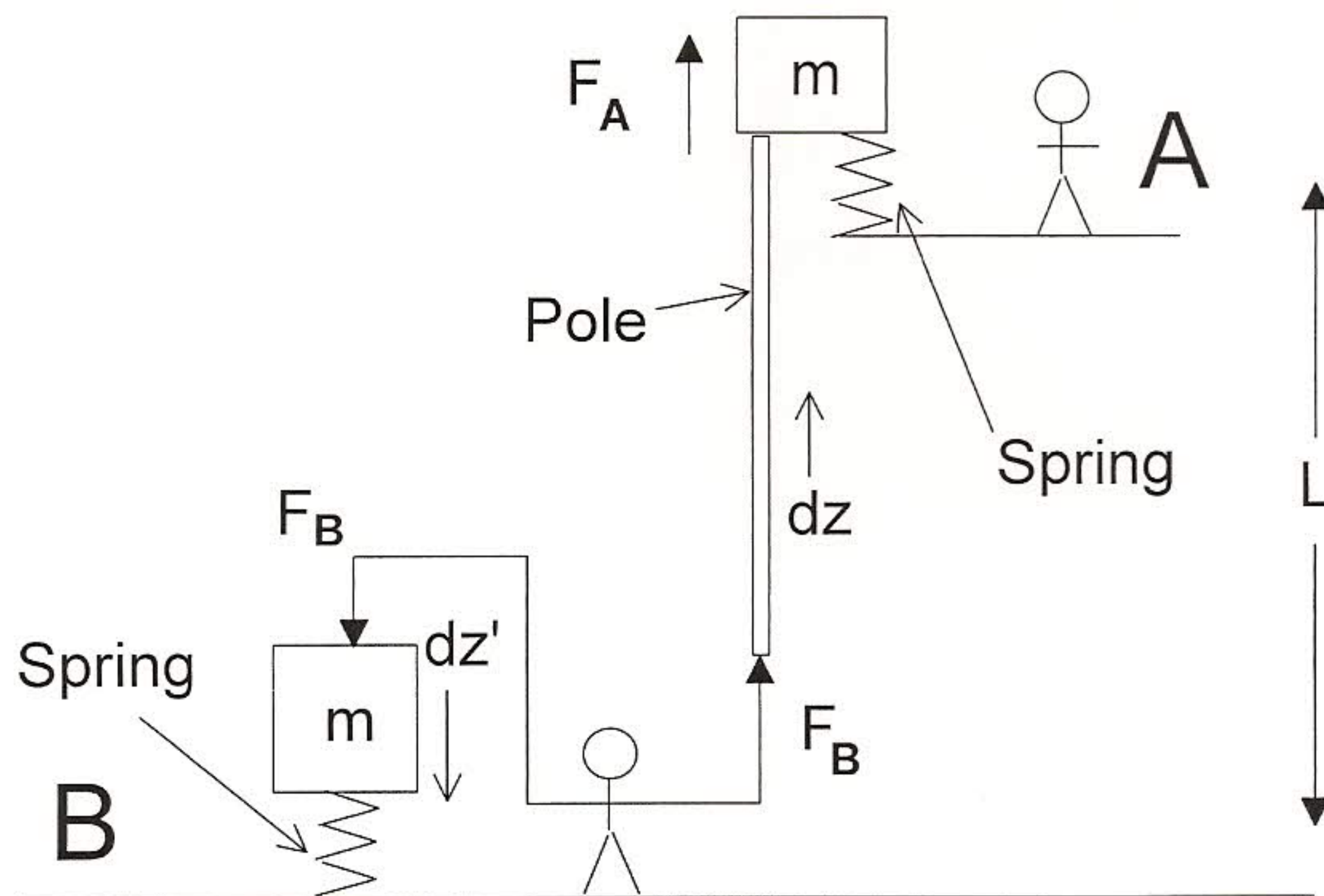


Figure 16

