

The article *The Real Ladder Paradox* showed how the acceleration of an object with length can be influenced by the Law of Conservation of Energy. This should be just a geometry analysis, so why is energy involved? This is an indication that energy and geometry are linked at some fundamental level. To explore this idea further, the length contraction process during acceleration will be examined in detail.

Definition of Terms

Figure 8 shows an object accelerating relative to inertial reference frames i and o.

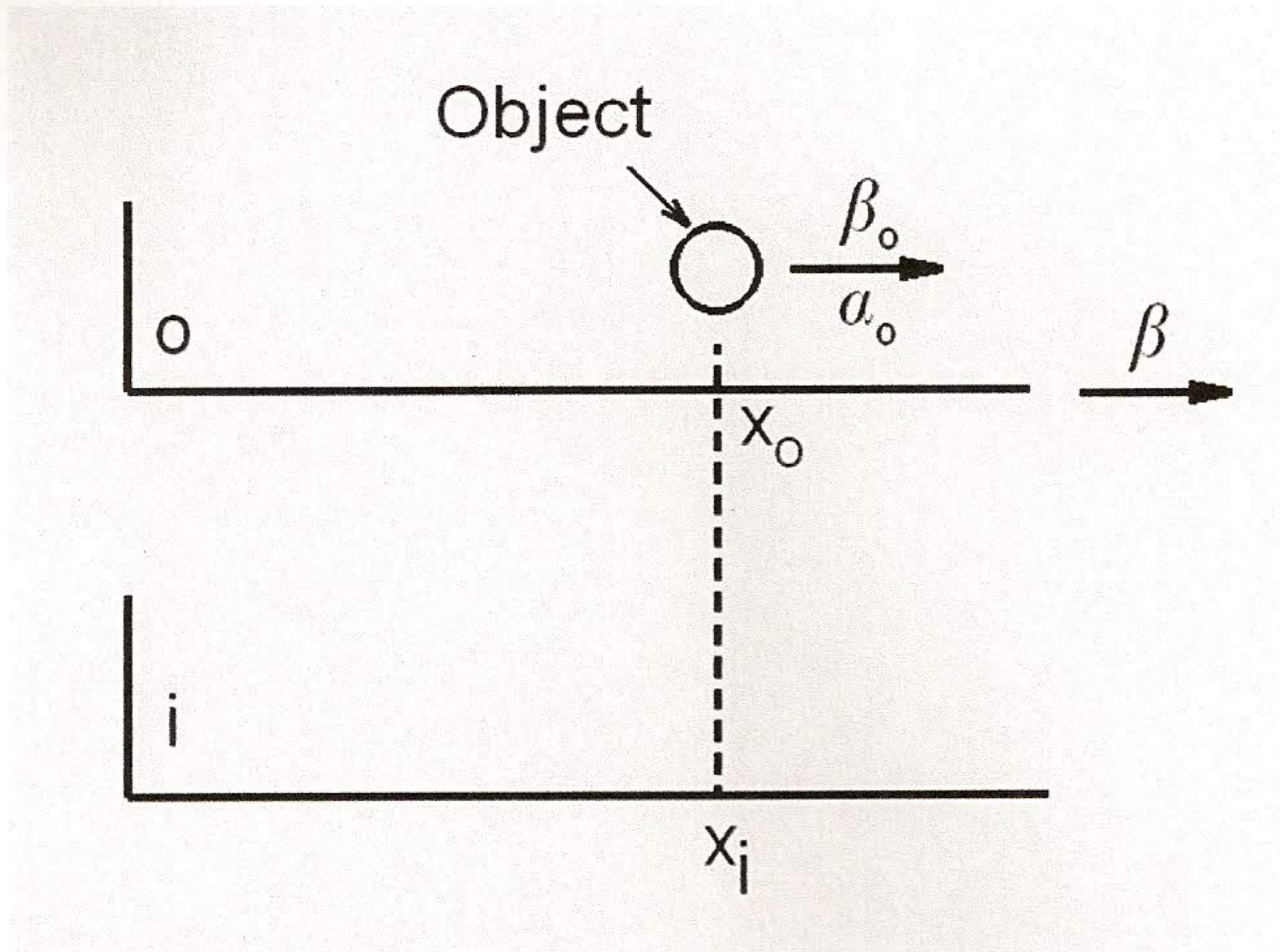


Figure 8. Single point object accelerates relative to frame i.

Frame o is moving with velocity β relative to frame i. Constant acceleration α_o is 'felt' by observers on the object. The acceleration starts from coordinate zero in frame i and at time zero for clocks on the object and both reference frames. The speed of light is c. The acceleration of the object relative to frame i can be found from the familiar velocity addition formula of Special Relativity. If β_i and β_o are the relativistic velocities of the object relative to frame i and o, then:

$$\beta = \frac{\text{velocity}}{c}$$

$$\alpha = \frac{\text{acceleration}}{c}$$

$$\beta_i = \frac{\beta + \beta_o}{1 + \beta\beta_o} \quad (9)$$

Differentiating (9) gives:

$$\frac{d\beta_i}{dt_i} = \frac{d\beta_o}{dt_o} \left[\frac{1 - \beta^2}{(1 + \beta\beta_o)^2} \right] \quad (10)$$

If t_i and t_o are clock readings and x_i and x_o are coordinates in frame i and frame o, then:

$$t_i = \frac{t_o + x_o\beta/c}{\sqrt{1 - \beta^2}} \quad (11)$$

Knowing that $c\beta_o = dx_o/dt_o$, differentiating (11) gives:

$$\frac{dt_i}{dt_o} = \frac{1 + \beta\beta_o}{\sqrt{1 - \beta^2}} \quad (12)$$

Since $\alpha = \frac{d\beta}{dt}$, combining (10) and (12) gives:

$$\alpha_i = \frac{\alpha_o (1 - \beta^2)^{3/2}}{(1 + \beta\beta_o)^3} \quad (13a)$$

$$\alpha = \alpha_o (1 - \beta^2)^{3/2} \quad (13b)$$

Equation (13a) is the general result for the acceleration seen by frame i. However, the interesting case is when the object is stationary in frame o ($\beta_o = 0$). This is (13b). The subscript 'i' will now be dropped for quantities measured in frame i and the frame i acceleration is denoted as α to indicate this is a general observed acceleration of any object traveling with speed β .

When an object undergoes a constant proper acceleration α_o , (13b) can now be used to generate general expressions for quantities seen by frame i. The clock reading in frame i will be t . Assuming the object starts accelerating in frame i from coordinate zero at time zero, velocity as a function of time is:

$$\beta = \frac{\alpha_o t}{\sqrt{1 + (\alpha_o t)^2}} \quad (14a)$$

$$\alpha_o t = \frac{\beta}{\sqrt{1 - \beta^2}} \quad (14b)$$

The distance traveled by the object is x and is determined from the relation $c\beta = \frac{dx}{dt}$.

$$x = \frac{c}{\alpha_o} \left[\sqrt{1 + (\alpha_o t)^2} - 1 \right] \quad (15)$$

The clock reading on the object is t_o . It is found from the relation $dt_o = dt \sqrt{1 - \beta^2}$.

$$t_o = \frac{1}{\alpha_o} \ln \left[\alpha_o t + \sqrt{1 + (\alpha_o t)^2} \right] \quad (16)$$

Definition of the proper acceleration model

The acceleration of a rod (object with length) will be modeled as an acceleration of two independent single point objects placed at the ends of the rod. Assume these two objects start at coordinates zero and L in inertial frame i , and they accelerate identically and simultaneously to a given velocity β . Each object will travel a distance x as given by (15). At the end of the acceleration, frame i will see each object arrive simultaneously at velocity β then stop accelerating. Frame i will still see them a distance L apart.

Now assume that a second inertial reference frame ii is traveling by at velocity β . This

reference frame will see the objects a distance $\frac{L}{\sqrt{1 - \beta^2}}$ apart after they finish accelerating. The rod that connects the objects must stretch to accommodate their stationary positions in frame ii . The total energy required to accelerate the rod will be the kinetic energy required for the change in velocity and additional energy to stretch the rod. The hypothesis is now stated that the proper acceleration model for an object with length does not involve any stretching or compressing of the object during the acceleration. This model will require that only the kinetic energy necessary for the velocity change be expended. Since a rod with identical accelerations at each end will stretch during acceleration, this means that the proper acceleration model will require the objects at the ends of the rod to have different accelerations as the rod changes velocity.

Accelerating an object with length

The object with length is two single point objects A and B at the ends of a rod. See Figure 9.

Object B has constant acceleration α_B and object A has constant acceleration α_A as felt by observers on the objects. Object B starts accelerating from coordinate zero in frame i and object A starts from coordinate L . Frame i sees both objects accelerate to velocity β , but it takes each a different amount of time because each has a different acceleration. Both objects and frame i start the experiment at clock readings of zero. The frame i time for object A and object B to get to velocity β is t_A and t_B .

$$\alpha_A t_A = \alpha_B t_B = \frac{\beta}{\sqrt{1 - \beta^2}} \quad (17)$$

In addition, if x_A is the distance traveled by object A as it gets to velocity β and x_B is the distance traveled by object B as it gets to velocity β , then:

$$\alpha_A x_A = \alpha_B x_B \quad (18)$$

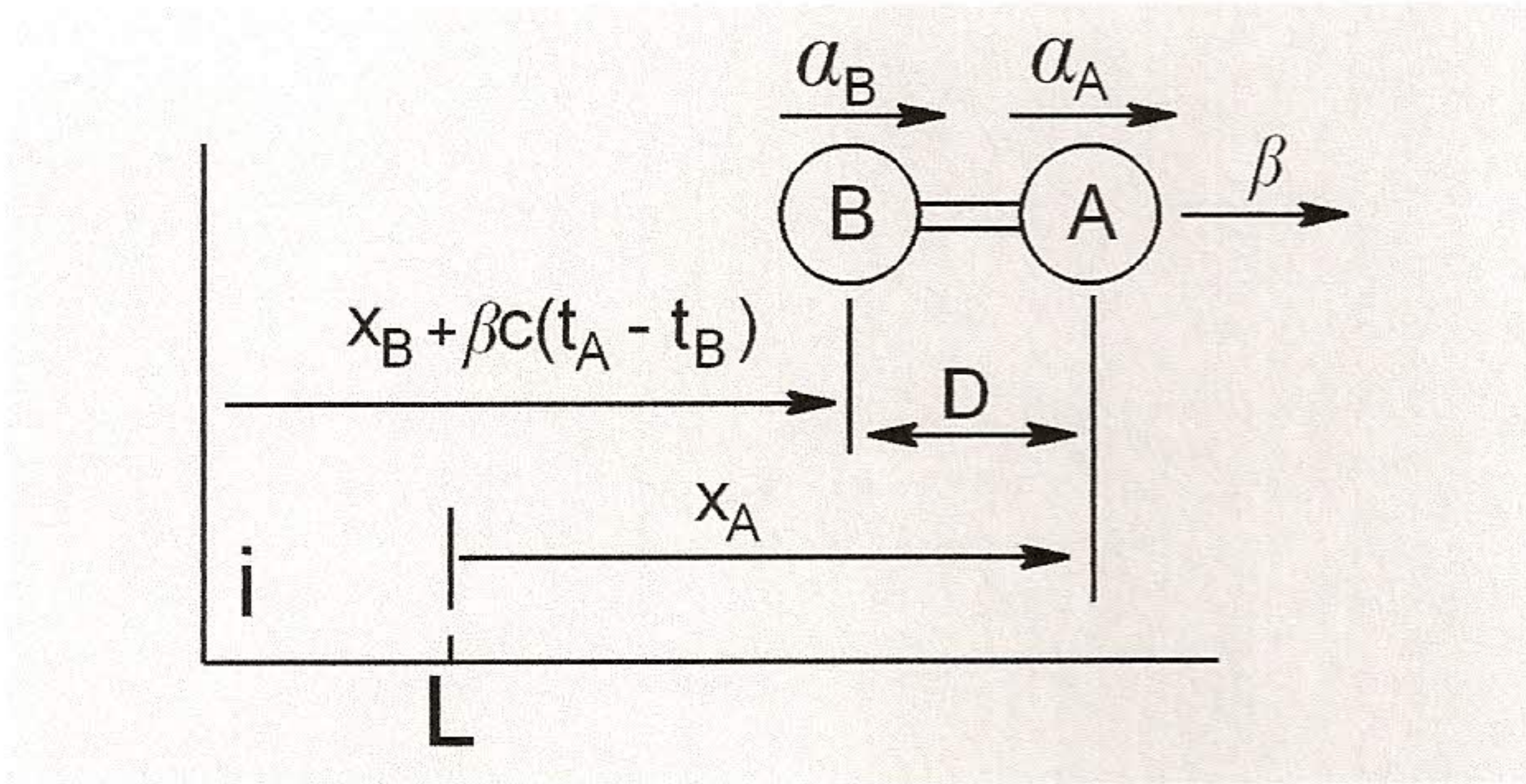


Figure 9. Acceleration of rod with single point objects A and B at the ends.

At the instant each object reaches any velocity β , the object clocks read t_{oA} and t_{oB} .

$$\alpha_A t_{oA} = \alpha_B t_{oB} \quad (19)$$

With this information, the length contraction process during acceleration can be examined. Both objects start accelerating simultaneously, but frame i sees object B finish first at time t_B . From that point forward, object B will travel with constant velocity β . Object A finishes accelerating at time t_A . In the time interval $t_A - t_B$, object B will travel distance $\beta c(t_A - t_B)$. Two separate equations can be written for the length D that frame i sees for rod length when it reaches velocity β .

$$D = L\sqrt{1 - \beta^2} \quad (20a)$$

$$D = (x_A + L) - (x_B + \beta c(t_A - t_B)) \quad (20b)$$

Solving (20):

$$L\sqrt{1-\beta^2} - L = \left(\frac{\alpha_B}{\alpha_A} - 1 \right) (x_B - \beta ct_B)$$

$$x_B = \frac{c}{\alpha_B} \left(\sqrt{1 + (\alpha_B t_B)^2} - 1 \right) = \frac{c}{\alpha_B} \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)$$

$$\beta ct_B = \frac{c}{\alpha_B} \left(\frac{\beta^2}{\sqrt{1-\beta^2}} \right)$$

$$L = \frac{c}{\alpha_B} \left(\frac{\alpha_B}{\alpha_A} - 1 \right)$$

$$\alpha_A = \frac{\alpha_B}{1 + \frac{\alpha_B L}{c}} \quad (21)$$

If object A has acceleration α_A as given by (21), then the rod connecting the two objects will not be stretched or compressed from its original length L when it becomes stationary in frame ii. This is the Special Relativity Law of Acceleration.

View of experiment from reference frame ii

The experiment of Figure 9 will now be shown from the viewpoint of reference frame ii. See Figure 10, where frame ii is stationary and frame i is passing by with velocity β in the minus direction. The two origins coincide in the view shown by Figure 10 when both origin clocks read zero. The coordinate L in frame i is opposite coordinate $L\sqrt{1-\beta^2}$ in frame ii, but this is not the starting point of object A in the experiment. The clock readings of both objects begin the experiment with a reading of zero in frame i. But, because frame ii sees object A trailing object B, its clock reading at the moment shown is not zero. Due to 'failure of simultaneity at a distance', frame ii sees the experiment start when the object A clock reads zero at frame ii time $t_{ii} = \frac{-L\beta}{c\sqrt{1-\beta^2}}$. At this point, object A is located at frame ii coordinate $x_{ii} = \frac{L}{\sqrt{1-\beta^2}}$. The view shown in Figure 10 is not quite correct at the instant where the origins coincide, as object A has already started accelerating and is no longer at coordinate L .

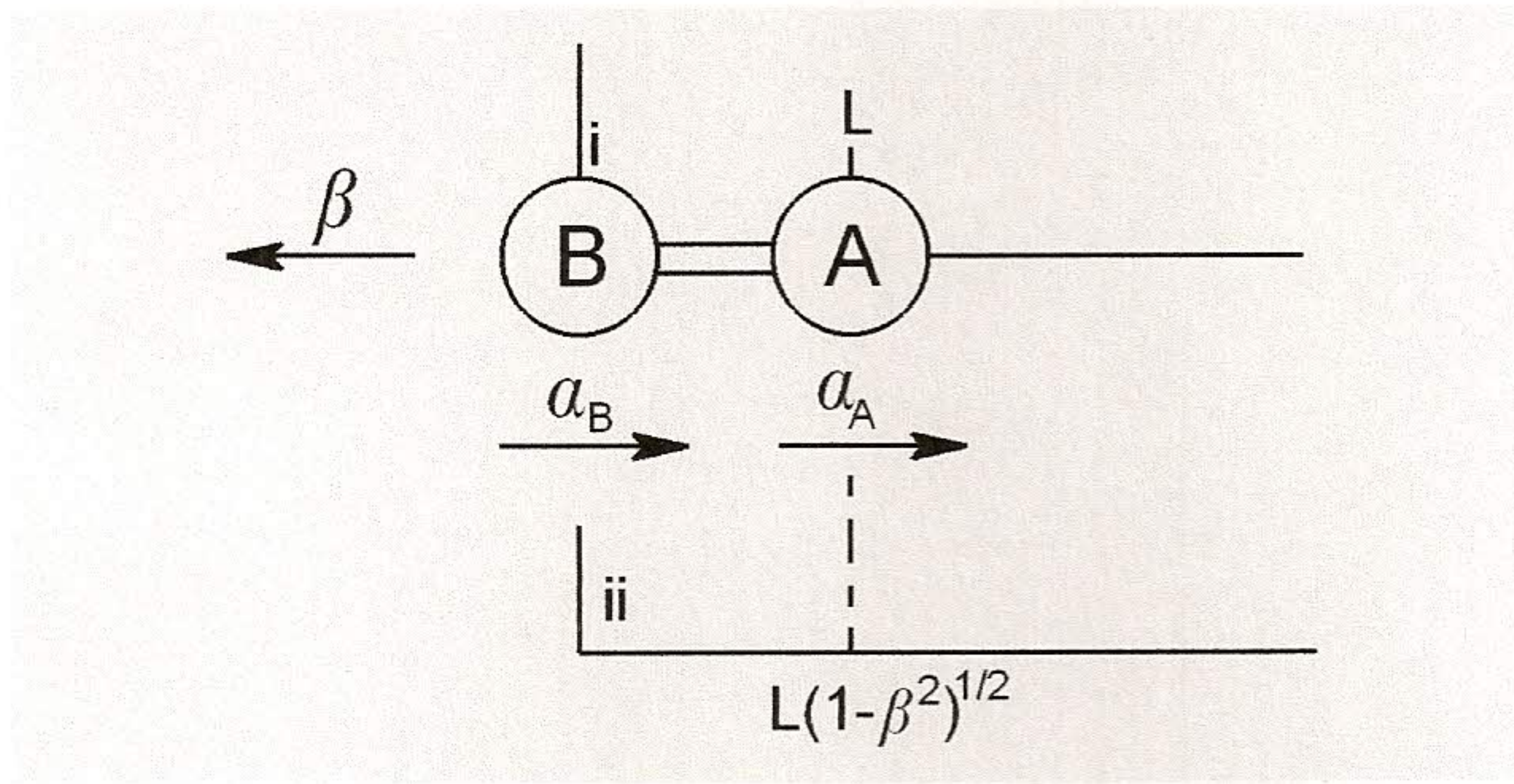


Figure 10. Frame ii view of experiment, around the start of the accelerations.

If a single object feels constant acceleration α_o , frame ii will see acceleration α_{ii} .

$$\alpha_{ii} = \alpha_o (1 - \beta^2)^{3/2} \quad (22)$$

This equation is used to derive expressions for the velocities β_{iiA} and β_{iiB} of objects A and B as seen by frame ii. Before considering the thought experiment where frame ii watches objects A and B decelerate from frame i, consider the case where the objects are initially stationary in frame ii and accelerate to velocity β . Object A would require time period $t_{ii} = t_A$ and object B would require time period $t_{ii} = t_B$. So, decelerating from $-\beta$ to zero at the same acceleration would also take these same time periods. When frame ii watches the deceleration of the objects,

object A has a velocity of $-\beta$ at time $t_{ii} = \frac{-L\beta}{c\sqrt{1-\beta^2}}$ and will have a velocity of zero at time $t_{ii} = t_A - \frac{L\beta}{c\sqrt{1-\beta^2}}$. Object B has a velocity of $-\beta$ at time $t_{ii} = 0$ and will have a velocity of zero at time $t_{ii} = t_B$. β_{iiA} and β_{iiB} are the velocities seen by frame ii at any time t_{ii} during the acceleration. To shorten the equations, dummy variable T is introduced.

$$T_A = t_A - \frac{L\beta}{c\sqrt{1-\beta^2}} - t_{ii}$$

$$\beta_{iiA} = \frac{-\alpha_A T_A}{\sqrt{1 + (\alpha_A T_A)^2}} \quad (23a)$$

$$\beta_{iiB} = \frac{-\alpha_B T_B}{\sqrt{1 + (\alpha_B T_B)^2}} \quad (23b)$$

The coordinate positions of objects A and B are x_{iiA} and x_{iiB} . The distance needed to accelerate to velocity β is x_A by object A and x_B by object B. The initial position of object A in frame ii is $x_{iiA} = \frac{L}{\sqrt{1 - \beta^2}}$ at time $t_{ii} = \frac{-L\beta}{c\sqrt{1 - \beta^2}}$ and the final position is $x_{iiA} = -x_A + \frac{L}{\sqrt{1 - \beta^2}}$ at time $t_{ii} = t_A - \frac{L\beta}{c\sqrt{1 - \beta^2}}$. The initial position of object B is $x_{iiB} = 0$ at time $t_{ii} = 0$ and the final position is $x_{iiB} = -x_B$ at time $t_{ii} = t_B$. Therefore:

$$x_{iiA} = \frac{c}{\alpha_A} \sqrt{1 + (\alpha_A T_A)^2} - \frac{c}{\alpha_A} \sqrt{1 + (\alpha_A t_A)^2} + \frac{L}{\sqrt{1 - \beta^2}} \quad (24a)$$

$$x_{iiB} = \frac{c}{\alpha_B} \sqrt{1 + (\alpha_B T_B)^2} - \frac{c}{\alpha_B} \sqrt{1 + (\alpha_B t_B)^2} \quad (24b)$$

To find the clock readings on the objects as seen by frame ii, the starting point is the time rate equations for objects A and object B.

$$dt_{oA} = dt_{ii} \sqrt{1 - \beta_{iiA}^2} = \frac{dt_{ii}}{\sqrt{1 + (\alpha_A T_A)^2}} \quad (25a)$$

$$dt_{oB} = dt_{ii} \sqrt{1 - \beta_{iiB}^2} = \frac{dt_{ii}}{\sqrt{1 + (\alpha_B T_B)^2}} \quad (25b)$$

From (25), the clock readings of objects A and B are t_{oA} and t_{oB} .

$$t_{oA} = \frac{1}{\alpha_A} \ln \left[\alpha_A t_A + \sqrt{1 + (\alpha_A t_A)^2} \right] - \frac{1}{\alpha_A} \ln \left[\alpha_A T_A + \sqrt{1 + (\alpha_A T_A)^2} \right] \quad (26a)$$

$$t_{oB} = \frac{1}{\alpha_B} \ln \left[\alpha_B t_B + \sqrt{1 + (\alpha_B t_B)^2} \right] - \frac{1}{\alpha_B} \ln \left[\alpha_B T_B + \sqrt{1 + (\alpha_B T_B)^2} \right] \quad (26b)$$

These equations can be used to show the exact states of objects A and B at any frame ii clock reading t_{ii} . Before doing this, (17) will be reconfigured.

$$\alpha_A t_A = \left(\frac{\alpha_B}{1 + \frac{\alpha_B L}{c}} \right) t_A = \frac{\beta}{\sqrt{1 - \beta^2}}$$

$$\alpha_B t_A = \frac{\beta}{\sqrt{1 - \beta^2}} + \frac{\alpha_B L \beta}{c \sqrt{1 - \beta^2}} = \alpha_B t_B + \frac{\alpha_B L \beta}{c \sqrt{1 - \beta^2}}$$

$$t_B = t_A - \frac{L \beta}{c \sqrt{1 - \beta^2}} \quad (27)$$

The clock reading $t_{ii} = t_B$ is the same as the clock reading $t_{ii} = t_A - \frac{L \beta}{c \sqrt{1 - \beta^2}}$. Object A reaches velocity zero in frame ii at the same instant as object B. Putting these clock readings into (24) gives the coordinate positions of object A and object B at the instant they become stationary in frame ii. The difference in these coordinate positions is the length of the rod in frame ii.

$$length = \left[\frac{c}{\alpha_A} - \frac{c}{\alpha_A} \sqrt{1 + (\alpha_A t_A)^2} + \frac{L}{\sqrt{1 - \beta^2}} \right] - \left[\frac{c}{\alpha_B} - \frac{c}{\alpha_B} \sqrt{1 + (\alpha_B t_B)^2} \right] \quad (28)$$

Simplifying (28):

$$length = \frac{c}{\alpha_A} - \frac{c}{\alpha_B} + \frac{L}{\sqrt{1 - \beta^2}} - \left(\frac{c}{\alpha_A} - \frac{c}{\alpha_B} \right) \sqrt{1 + (\alpha_A t_A)^2}$$

$$length = \frac{L}{\sqrt{1 - \beta^2}} - L \left[\left(1 + \frac{\beta^2}{1 - \beta^2} \right)^{1/2} - 1 \right]$$

$$length = L \quad (29)$$

Objects A and B become stationary in frame ii simultaneously a distance L apart. This satisfies the requirement of Special Relativity that there are no primary reference frames. Frame ii and

