Forces fall into three main categories: static, dynamic and field generated. The study of dynamic force begins with it's relationship to momentum.

The General Case

Consider the general case of a force impressed on a mass m. Relative to a stationary reference frame B, this force results in a change in the vector momentum $\vec{P}_B$ of the mass.

$$\vec{P}_B = \frac{mc\,\vec{\beta}_B}{\sqrt{1-\left|\vec{\beta}_B\right|^2}}$$

$$\vec{P}_B = P_{XB}\,\vec{i} + P_{YB}\,\vec{j} \qquad \vec{\beta}_B = \beta_{XB}\,\vec{i} + \beta_{YB}\,\vec{j}$$

$\vec{i}, \vec{j}$ - unit vectors in the x, y directions

$F_{XB} = dP_{XB} / dt_B$ - force on mass m in the x-direction

$F_{YB} = dP_{YB} / dt_B$ - force on mass m in the y-direction

$$F_{XB} = \frac{mc}{\left(1-\beta_{XB}^2-\beta_{YB}^2\right)^{3/2}}\left[\left(1-\beta_{YB}^2\right)A_{XB} + \beta_{XB}\beta_{YB}A_{YB}\right] \qquad (20a)$$

$$F_{YB} = \frac{mc}{\left(1-\beta_{XB}^2-\beta_{YB}^2\right)^{3/2}}\left[\left(1-\beta_{XB}2\right)A_{YB} + \beta_{XB}\beta_{YB}A_{XB}\right] \qquad (20b)$$

$A_{XB} = \dfrac{d\beta_{XB}}{dt_B}$ - (x-direction acceleration of m)/c

$A_{YB} = \dfrac{d\beta_{YB}}{dt_B}$ - (y-direction acceleration of m)/c

The "Force Enhancement" Principle of Special Relativity

Equation (20) can be applied to experiments where force and momentum are involved. For example, consider the experiment of Figure 7, where a stationary reference frame B is pushing on a mass m with forces F_{XB} and F_{YB}. Mass m is stationary in a reference frame A that is moving with x-direction velocity β_X (the y-direction velocity is $\beta_Y = 0$).

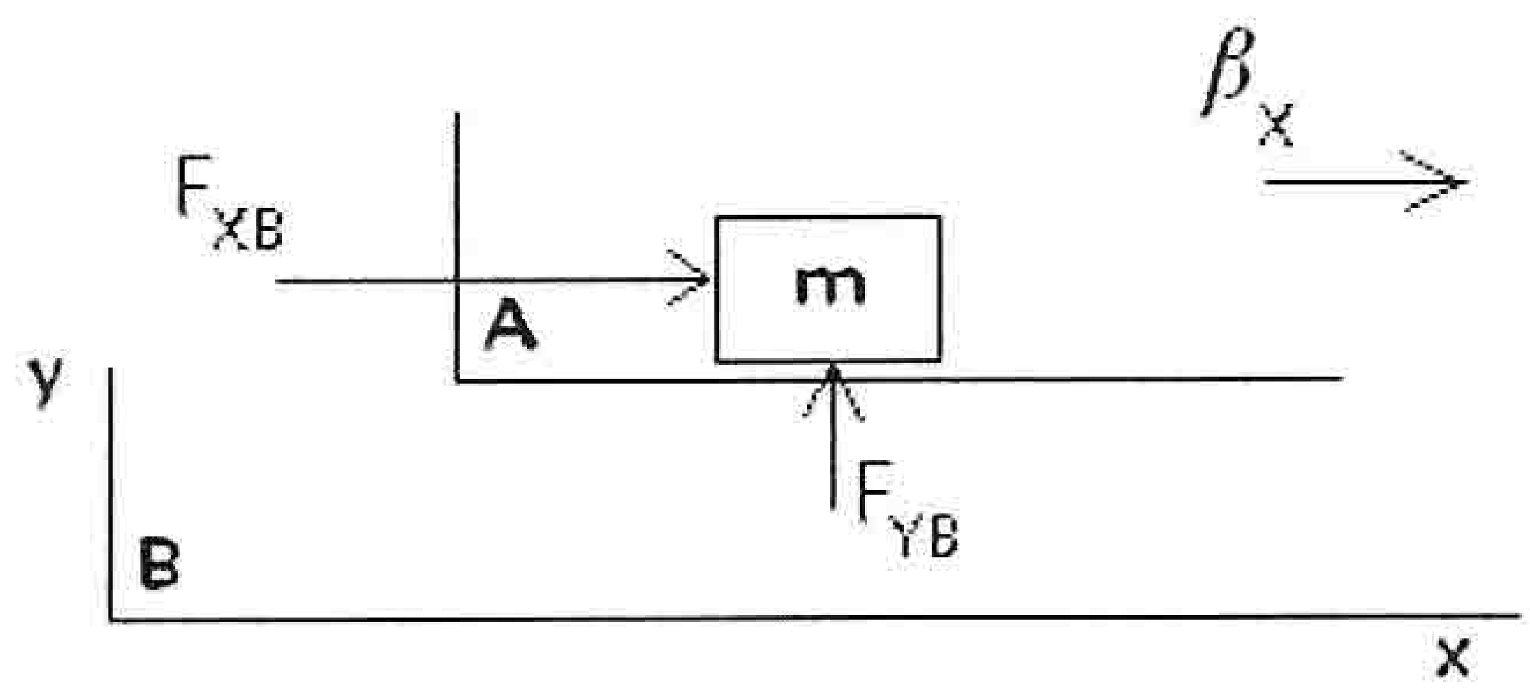


Figure 7. A simple force experiment.

Although the experiment of Figure 7 appears simple, it lacks critical information. The forces F_{XB} and F_{YB} are stationary in frame B but are applied to a moving mass. The analysis of any force experiment will require that the mechanisms of force application be well defined. Force F_{XB} could be applied by a spring mounted to frame B and pressing on moving mass m. But force F_{YB} requires a mechanism that can push perpendicular to the direction of the velocity of mass m. One way to do this is shown in Figure 8.

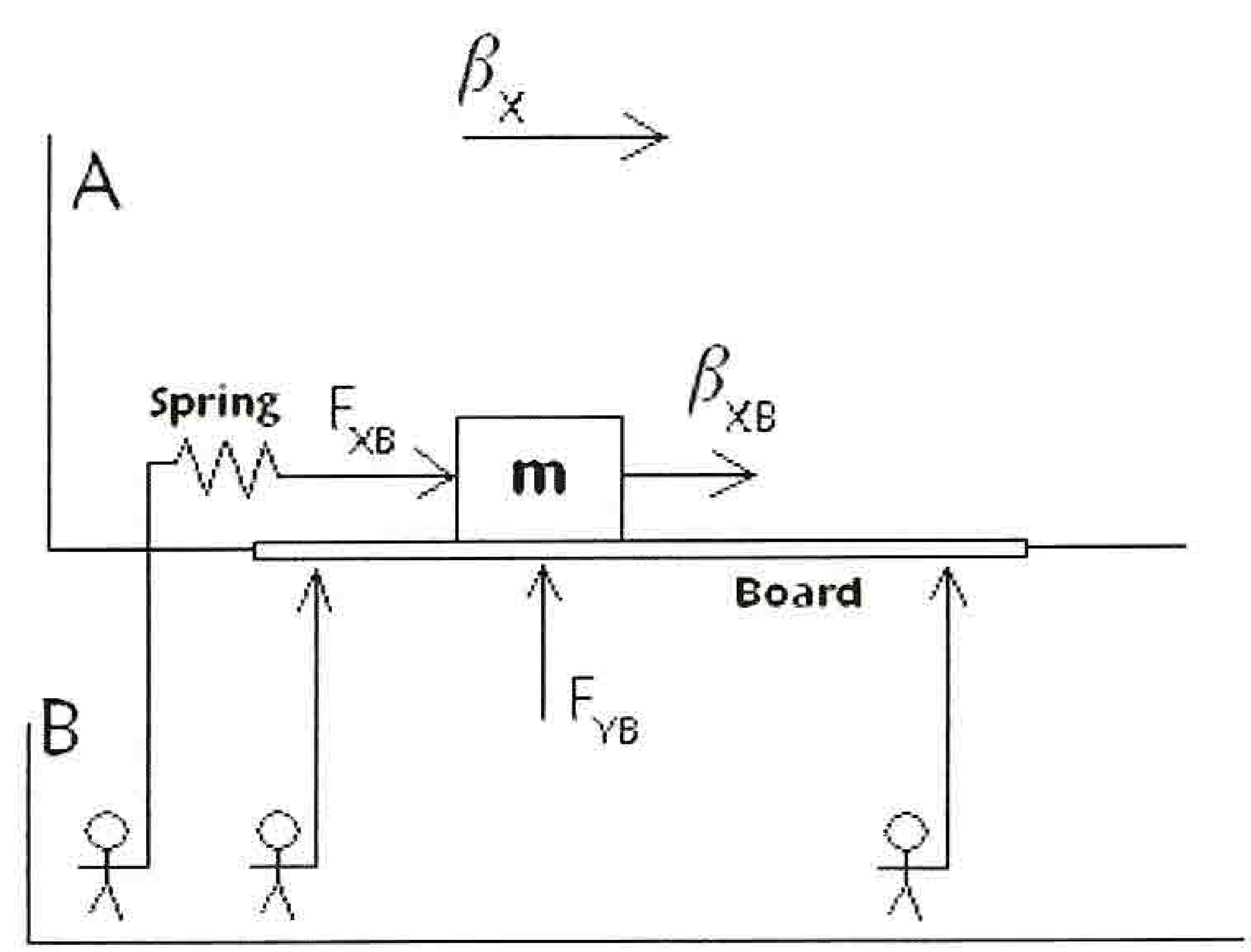


Figure 8. The experiment of Figure 7 with added details of force application.

In Figure 8, two observers are pushing vertically on each end of a board with a frictionless surface that mass m slides across. The observers are careful to keep the board parallel to the x-axis at all times and they are careful to make sure that the total force on mass m is a constant F_{YB}. Mass m is moving with velocity $\beta_{XB} = \beta_X$ and $\beta_{YB} = \beta_y = 0$ relative to frame B (mass m is stationary in frame A). No momentum change of frame B occurs, as a mirror image experiment is run simultaneously in the opposite direction , but not shown.

It is desired to find the relationship between the forces applied by frame B and the forces that mass m "feels", which would be the forces measured by a reference frame in which mass m is stationary. This happens to be frame A. Applying (20):

$$F_{XB} = \frac{mcA_{XB}}{\left(1-\beta_{XB}^2\right)^{3/2}} \tag{21a}$$

$$F_{YB} = \frac{mcA_{YB}}{\sqrt{1-\beta_{XB}^2}} \tag{21b}$$

Applying the conditions of this experiment to (16) and (17):

$$A_{XB} = A_{XA}\left(1-\beta_{XB}^2\right)^{3/2} \tag{22a}$$

$$A_{YB} = A_{YA}\left(1-\beta_{XB}^2\right) \tag{22b}$$

The forces that mass m "feels" are the forces observed in reference frame A. These forces are $F_{XA} = mcA_{XA}$ and $F_{YA} = mcA_{YA}$. The final result is:

$$F_{XB} = F_{XA} \tag{23a}$$

$$F_{YB} = F_{YA}\sqrt{1-\beta_{XB}^2} \tag{23b}$$

The result (23) gives transformations for forces applied between inertial references frames where mass m is stationary in frame A. Equation (23) is a principle of Special Relativity. It joins the other three principles (length contraction, time dilation and failure of simultaneity at a distance) which explain the fundamental workings of Special Relativity. This principle will be called "Force Enhancement", due to the way y-directed forces are increased when they pass from the stationary frame B to the moving frame A.

Geometry Distortion and Force Application

The case where there is a frame A y-direction velocity to mass m is worthy of investigation. In Figure 9, the experiment is the same as that of Figure 8 except that mass m is moving with velocity $\vec{\beta}_B$. Also note that force F_{XB} has now been set to zero.

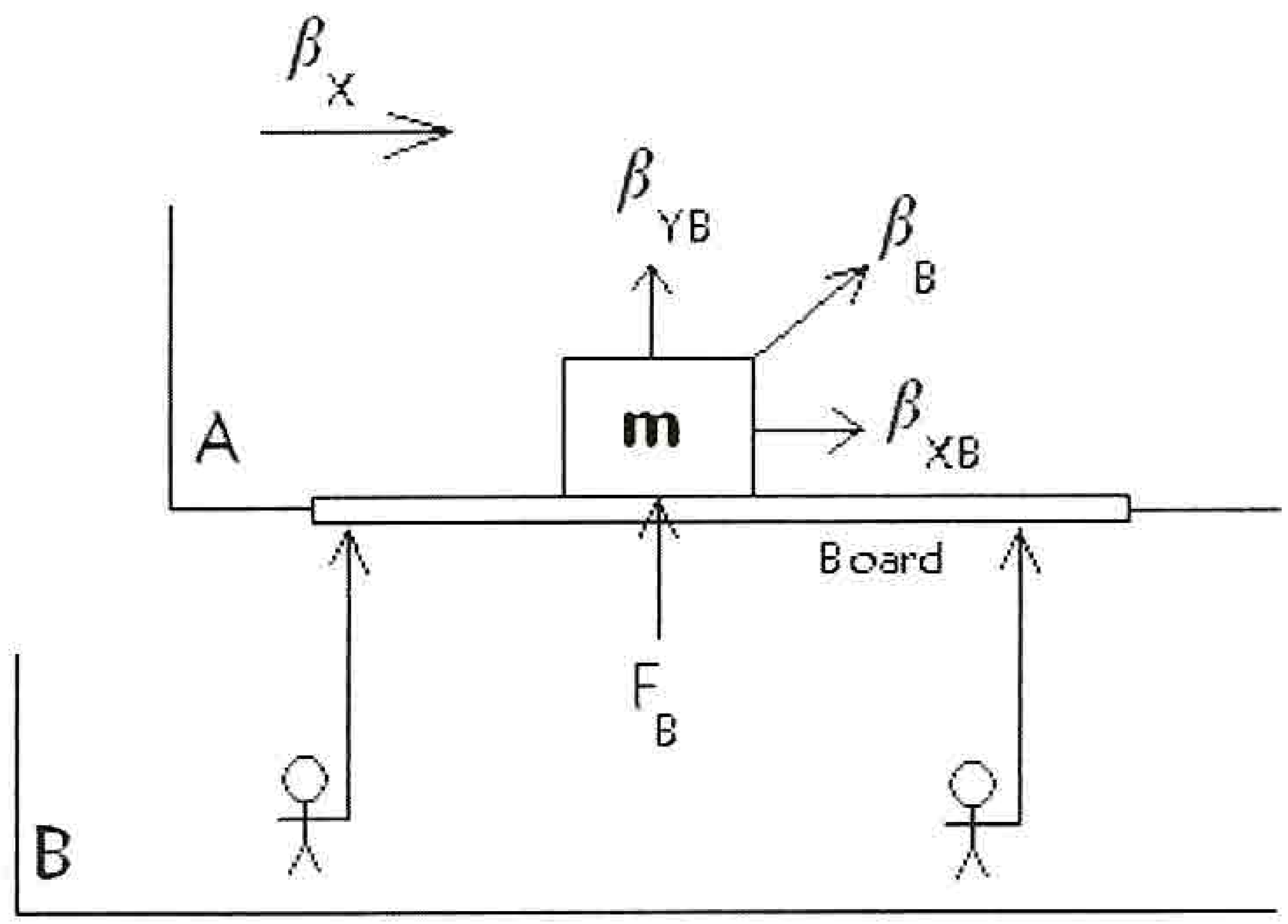


Figure 9. Experiment with mass m moving in frame A.

Frame B sees mass m move in response to any impressed force as described by (20). In this case, (20a) gives:

$$A_{XB} = -\frac{\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2} \tag{24}$$

The frame B observers see an acceleration of mass m in the x-direction, even though the experiment is specifically structured without an x-directed force. They also see mass m move vertically in response to their impressed force F_B. Inserting (24) into (20b) gives the reaction of mass m as seen by frame B.

$$F_B = \frac{mcA_{YB}}{\sqrt{1-\beta_{XB}^2-\beta_{YB}^2}\left(1-\beta_{YB}^2\right)} \tag{25}$$

To assist in analyzing what happens in frame A, frame M in which mass m is stationary will be added to the experiment. Frame M moves upward at velocity β_{YA} relative to frame A, as is shown in Figure 10.

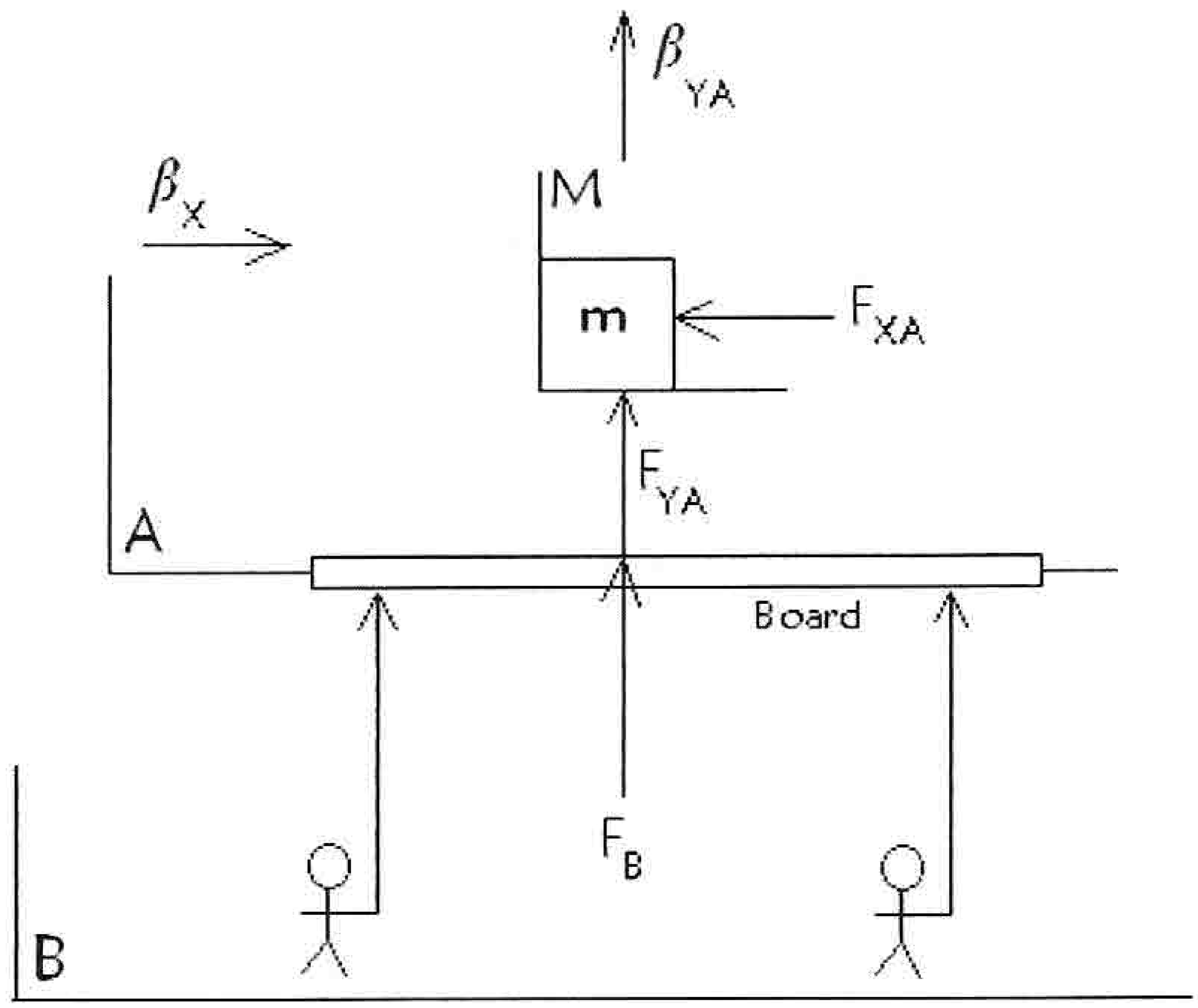


Figure 10. Mass m is stationary in reference frame M.

In Figure 10, the frame A velocity is $\beta_X = \beta_{XB}$, so that $\beta_{XA} = 0$ for mass m. Mass m is stationary in frame M at the instant shown. It is now known from (24) that frame A sees a "mysterious force" F_{XA} applied to mass m. How the board applies this force will be explained later. For now, the relationship of the forces in Figure 10 can be written down by adapting (20) to the frame A point of view.

$$F_{YA} = \frac{mcA_{YA}}{\left(1 - \beta_{YA}^2\right)^{3/2}} \tag{26a}$$

$$F_{XA} = \frac{mcA_{XA}}{\sqrt{1 - \beta_{YA}^2}} \tag{26b}$$

The relationship between β_{YA} and β_{YB} can be found from (13).

$$\beta_{YA} = \frac{\beta_{YB}}{\sqrt{1 - \beta_{XB}^2}} \tag{27}$$

The relationship between A_{XA} and A_{XB} can be found from (16).

$$A_{XB} = A_{XA}\left(1-\beta_{XB}^2\right)^{3/2} \tag{28}$$

The relationship between A_{YA} and A_{YB} can be found from (17) and (24),

$$A_{YA} = A_{YB}\frac{1-\beta_{XB}^2-\beta_{YB}^2}{\left(1-\beta_{XB}^2\right)^2\left(1-\beta_{YB}^2\right)} \tag{29}$$

Inserting (27) and (28) into (26b):

$$F_{XA} = \frac{mcA_{XB}}{\sqrt{1-\beta_{XB}^2-\beta_{YB}^2}\left(1-\beta_{XB}^2\right)} \tag{30}$$

Inserting (27) and (29) into (26a):

$$F_{YA} = \frac{mcA_{YB}}{\sqrt{1-\beta_{XB}^2-\beta_{YB}^2}\left(1-\beta_{YB}^2\right)\sqrt{1-\beta_{XB}^2}} \tag{31}$$

And:

$$\frac{F_{XA}}{F_{YA}} = \frac{-\beta_{XB}\beta_{YB}}{\sqrt{1-\beta_{XB}^2}} \tag{32}$$

In this example, there is a horizontal force observed by frame A, even though frame B only applies a vertical force. So, where does "mysterious force" F_{XA} come from? The answer is that frame B applies this force. The way this occurs is shown by viewing the experiment from the point of view of frame A, as is shown in Figure 11.

Frame A sees frame B going by with velocity $-\beta_{XB}$. The distance between the two observers impressing the vertical force on the board is L and frame A sees this distance as $L\sqrt{1-\beta_{XB}^2}$. The board has velocity β_{YB} as seen by frame B. The observer that is on the right in Figure 11 is observed by frame A to have a clock that reads "later" than the left observer clock by $L\beta_{XB}/c$. In this time period, the board travels distance $L\beta_{YB}\beta_{XB}$ as seen by either frame. So the right observer's section of the board has gone up further than the left observers section. This effect is engaged linearly down the length of the board so that the board is tilted as observed by frame A. The force applied by the board is perpendicular to the board surface as seen by either frame. Simple geometry gives:

$$\frac{F_{XA}}{F_{YA}} = \frac{-L\beta_{XB}\beta_{YB}}{L\sqrt{1-\beta_{XB}^2}} = \frac{-\beta_{XB}\beta_{YB}}{\sqrt{1-\beta_{XB}^2}} \qquad (33)$$

This is the same result as (32).

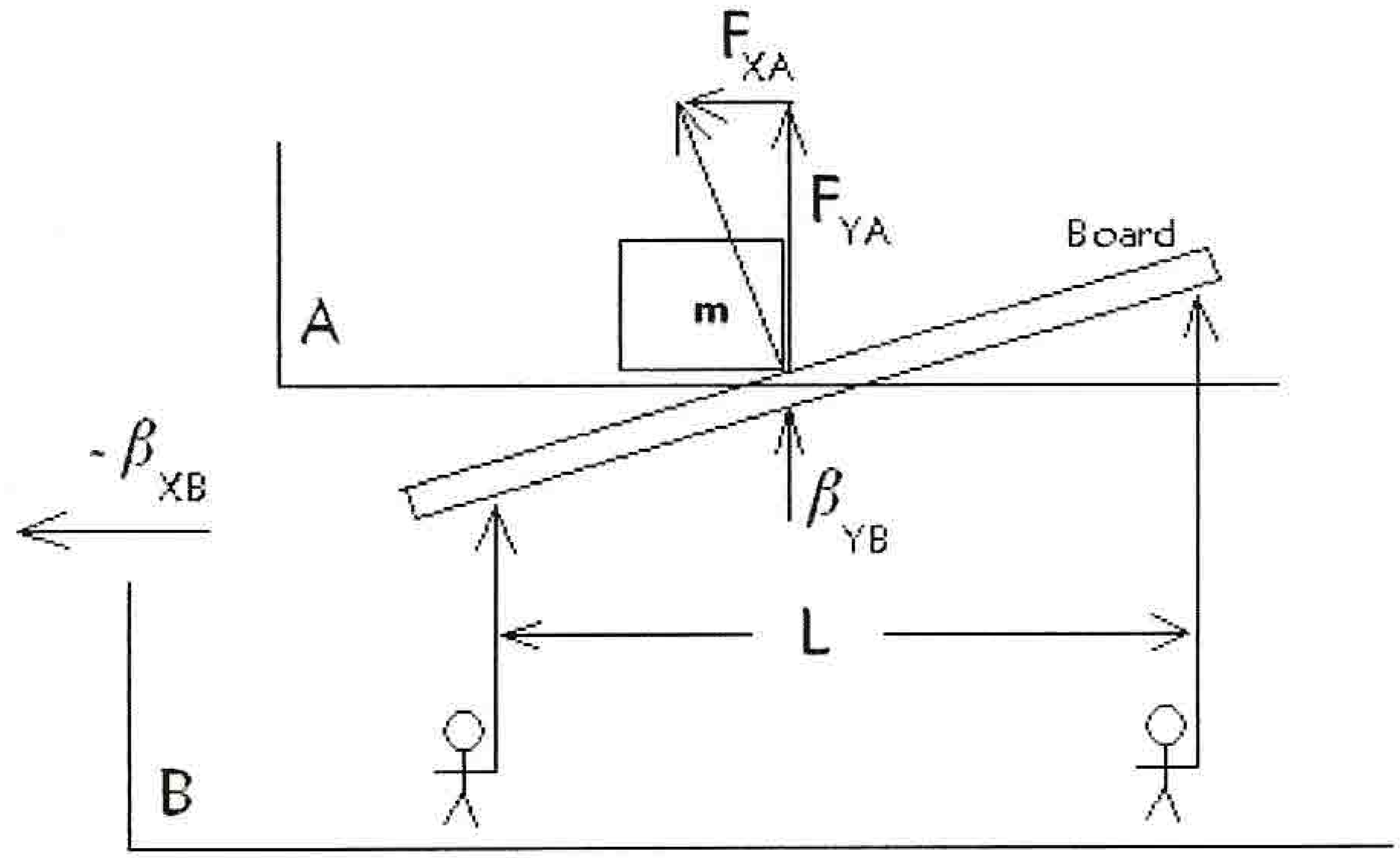


Figure 11. The experiment of Figure 10 as seen by reference frame A.

The space-time geometry of the experiment as viewed by frame A shows how a force perpendicular to the board surface can result in the appearance of a force in the x-direction. Because mass m is not stationary in frame A, this is only an approximate solution. It is reasonably accurate for low values of β_{XB} and β_{YB}, but loses accuracy as those velocities approach light speed. For an exact expression of (32), it is necessary to define frame A as one in which mass m is stationary. In that case:

$$\beta_X = \beta_{XB} \qquad \beta_Y = \beta_{YB} \qquad \beta_T^2 = \beta_X^2 + \beta_Y^2$$

$$F_{XA} = mcA_{XA} \qquad F_{YA} = mcA_{YA} \qquad (34)$$

Equation (24) can be inserted into factor K_B used in (16) and (17) to give:

$$K_B = A_{YB}\frac{\beta_Y - \frac{\beta_X\beta_{XB}\beta_{YB}}{1-\beta_{YB}^2}}{1-\beta_X\beta_{XB}-\beta_Y\beta_{YB}} \qquad (35)$$

And, using the definition of K_{TB} from (12) and (13):

$$S_1 = 1 - \frac{\beta_X^2}{\beta_T^2}\left(1 - \sqrt{1-\beta_T^2}\right) \qquad S_2 = 1 - \frac{\beta_X \beta_Y}{\beta_T^2}\left(1 - \sqrt{1-\beta_T^2}\right) \qquad S_3 = 1 - \frac{\beta_Y^2}{\beta_T^2}\left(1 - \sqrt{1-\beta_T^2}\right)$$

$$A_{XA} = \left[\left(\frac{-\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2} + \beta_{XB}K_B\right)S_1 - \left(A_{YB} + \beta_{YB}K_B\right)S_2\right]K_{TB}^2 \tag{36a}$$

$$A_{YA} = \left[\left(A_{YB} + \beta_{YB}K_B\right)S_3 - \left(\frac{-\beta_{XB}\beta_{YB}A_{YB}}{1-\beta_{YB}^2} + \beta_{XB}K_B\right)S_2\right]K_{TB}^2 \tag{36b}$$

$$\frac{F_{XA}}{F_{YA}} = \frac{A_{XA}}{A_{YA}} \tag{37}$$

Figure 12 shows the relationship of the board to mass m under these circumstances.

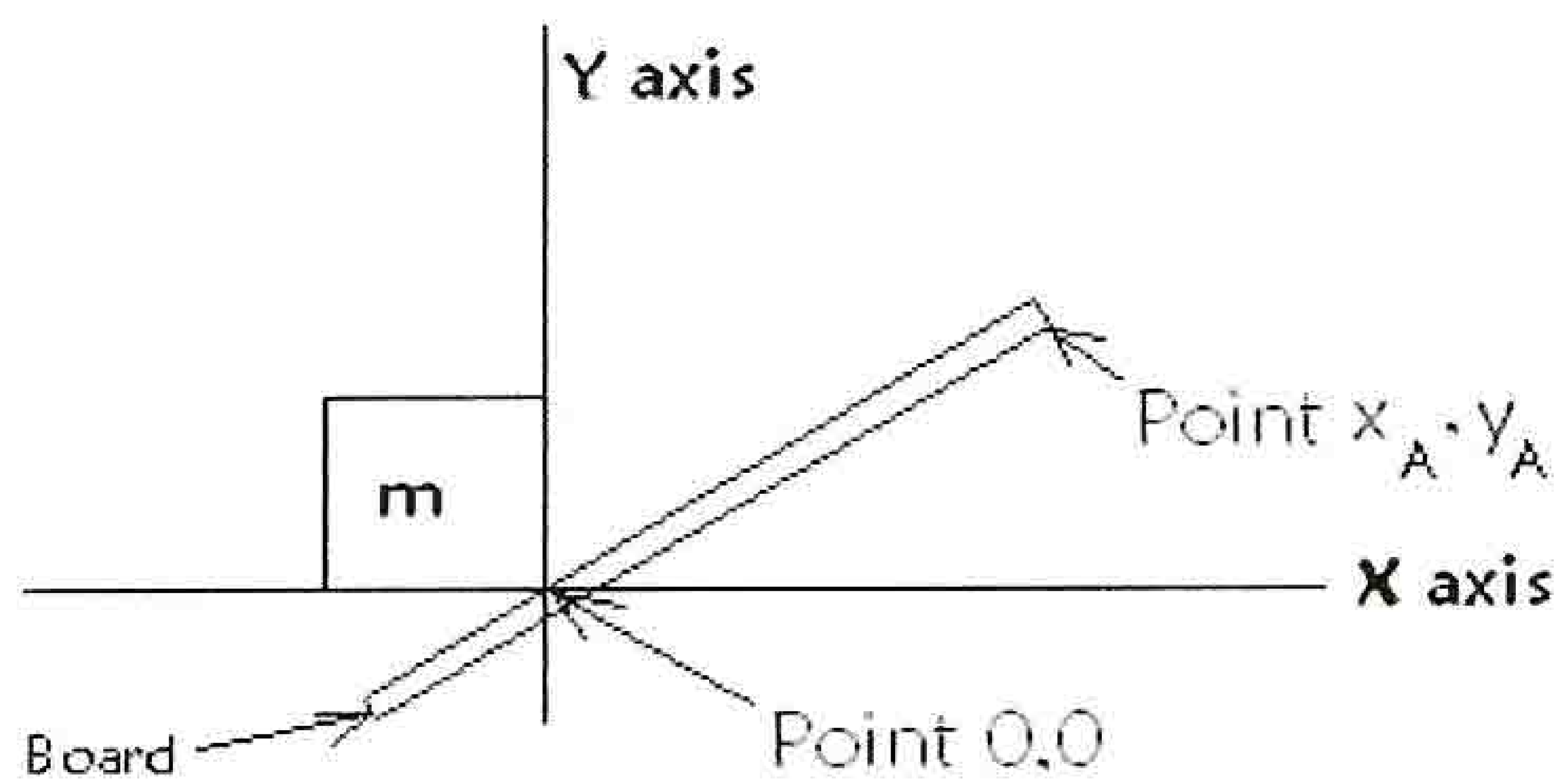


Figure 12. Orientation of the board as seen by frame A.

In Figure 12, frame A and frame B are instantaneously on top of each other with the origins coinciding at point 0,0. The board is oriented to give mass m a force in the x and y directions. In this position, point x_A, y_A on one end of the board describes the board's orientation. Equations (7) and (11) can be used to define the points shown at the instant $t_A = 0$ when mass m is instantaneously stationary in frame A. Under these conditions, $\beta_{XA} = \beta_{YA} = 0$. Point 0,0 is where the board touches mass m at the origins of both reference frames. Using (7), point 0,0 is located at the following coordinates:

$$x_A = y_A = x_B = y_B = 0 \tag{38}$$

Using (11) with $t_A = 0$, frame A sees the clock readings of the frame B coordinate points as:

$$t_B = \frac{x_B \beta_x}{c} + \frac{y_B \beta_Y}{c} \tag{39}$$

From the point of view of frame B, the following conditions apply at any coordinate point x_B, y_B along the board:

$$x_B = CONSTANT \qquad\qquad y_B = c\beta_{YB} t_B \tag{40}$$

Combining (40) with (39) will show how frame A sees the clock readings and board coordinates along the board.

$$t_B = \frac{x_B \beta_X}{c(1 - \beta_Y \beta_{YB})} \qquad\qquad y_B = \frac{x_B \beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}} \tag{41}$$

Now, (7) can be used to give:

$$x_A = x_B \left(S_1 - \frac{\beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}} S_2 \right) \tag{42a}$$

$$y_A = x_B \left(S_3 \frac{\beta_X \beta_{YB}}{1 - \beta_Y \beta_{YB}} - S_2 \right) \tag{42b}$$

Based upon simple geometry such as illustrated in Figure 11:

$$\frac{F_{XA}}{F_{YA}} = -\frac{y_A}{x_A} \tag{43}$$

Equations (36), (37) and (42) do not look as though they would reduce to (43). However, if these equations are put into a computer program to give numerical values to each of the quantities, it will be found that (43) is exactly true for all values of inputs. The geometric configuration of the board gives the exact ratio of forces to produce the movement of mass m as seen by frame B.

Because the equations leading up to (37) and (43) still have separate values for $\beta_X, \beta_Y, \beta_{XB}$ and β_{YB}, these equations can be used to determine the forces and board relationships for other reference frames moving with general velocities. These reference frames may have mass m moving within them or, by setting $\beta_X = \beta_{XB}$ and $\beta_Y = \beta_{YB}$, they will give results for the forces that mass m "feels" (mass m stationary in reference frame). If $\beta_Y = 0$ is used as an input, (37) and (43) will verify the accuracy of (33) and (32) at lower values of mass m velocity.

Practical Force Transformations

Though (37) and (43) may be used to solve problems within Special Relativity, a simpler method can transform forces between reference frames. Force transformations differ from length and time transformations in that forces are absolute, not relative between reference frames. If the frame B observer does an experiment where he applies a force of 7 Newtons to frame A and the frame A observer measures a force of 8 Newtons, these values do not change as observers switch frames.

Since force is absolute, (23) is absolute (not a relative transformation). Both reference frames see the same details of the force application - where the board is stationary in frame B and the mass is stationary in frame A. The locations of the board and mass determine the application of (23) to the experiment. Equation (23) is used by both reference frames in exactly the same form.

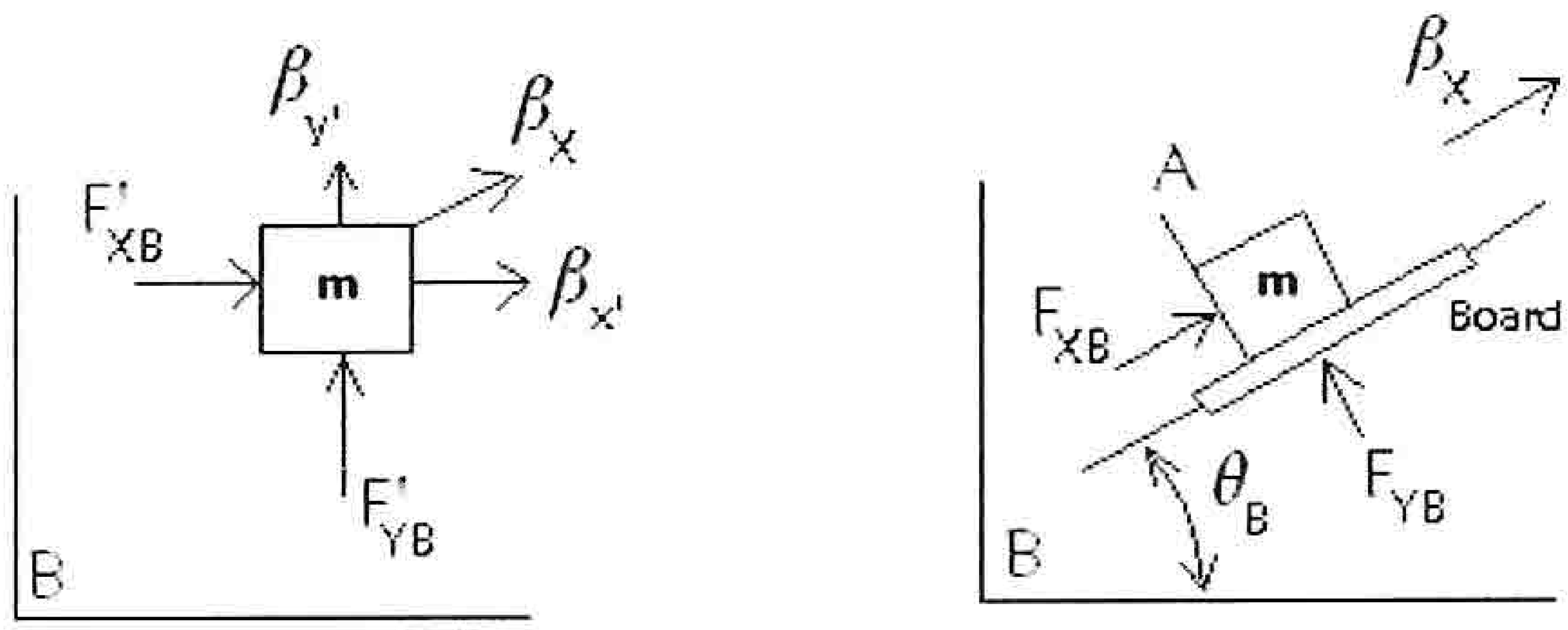
Applied forces exist between two specific points (two specific reference frames) and can only be measured in the reference frames in which they originate or terminate. All other reference frames can only calculate forces for the experiment. Other reference frames cannot "observe" these forces, they can only observe motions caused by the forces. As has been shown above, the measured accelerations and velocities of masses affected by forces can be transformed between various reference frames according to the equations in the article *Position, Velocity, Acceleration.* In addition, the forces on those masses can be calculated, using (20). But this is only of passing interest in most experiments. Only the reference frames where forces originate and terminate have any direct interest in how the forces transform.

A solution procedure can be stated for a general experiment illustrated in Figure 13. In Figure 13a, a mass m is moving in reference frame B with velocities $\beta_{X'}$ and $\beta_{Y'}$. These velocities resolve into a main velocity which shall be called β_X. Reference frame B is pushing on mass m with forces F'_{XB} and F'_{YB}. It is desired to find the forces that mass m feels.

The first step is shown in Figure 13b. Velocity β_X is defined as the x-direction velocity of reference frame A in which mass m is instantaneously stationary. Angle θ_B defines the orientation of β_X as seen by frame B. Using simple geometry:

$$F_{XB} = F'_{YB} \sin\theta_B + F'_{XB} \cos\theta_B \qquad (44a)$$

$$F_{YB} = F'_{YB}\cos\theta_B - F'_{XB}\sin\theta_B \tag{44b}$$



(a) (b)

Figure 13. General force experiment.

Using (44) and knowing that $\beta_X^2 = \beta_{X'}^2 + \beta_{Y'}^2$, (23) can be stated:

$$F_{XA} = \frac{F_{YB}\beta_{Y'} + F_{XB}\beta_{X'}}{\beta_X} \tag{45a}$$

$$F_{YA} = \frac{F_{YB}\beta_{X'} - F_{XB}\beta_{Y'}}{\beta_X\sqrt{1-\beta_X^2}} \tag{45b}$$

The simple solution (45) for a general force problem results directly from the fourth principle of Special Relativity (23).

The derivation of the frame A view of the experiment of Figure 13 is done in a similar manner. From frame A, the experiment appears as shown in Figure 14. In Figure14(a), mass m is stationary in frame A and the board is stationary in frame B. Frame B is moving with velocity $-\beta_X$. In Figure 14(b), the original frame B is replaced by a temporary reference frame called "New B". It also moves with velocity $-\beta_X$, but has it's x-axis parallel to the frame A x-axis (and β_X). "New B" applies forces F_{XB} and F_{YB} to mass m. Equation (23) applies directly to Figure 14(b). The forces that the original frame B applies to mass m are forces F'_{XB} and F'_{YB}. The original frame B sees "New B" as stationary and at an angle of θ_B. The geometry between "New B" and the original frame B gives:

$$F'_{XB} = \frac{F_{XA}\beta_{X'} - F_{YA}\beta_{Y'}\sqrt{1-\beta_X^2}}{\beta_X} \quad (46a)$$

$$F'_{YB} = \frac{F_{XA}\beta_{Y'} + F_{YA}\beta_{X'}\sqrt{1-\beta_X^2}}{\beta_X} \quad (46b)$$

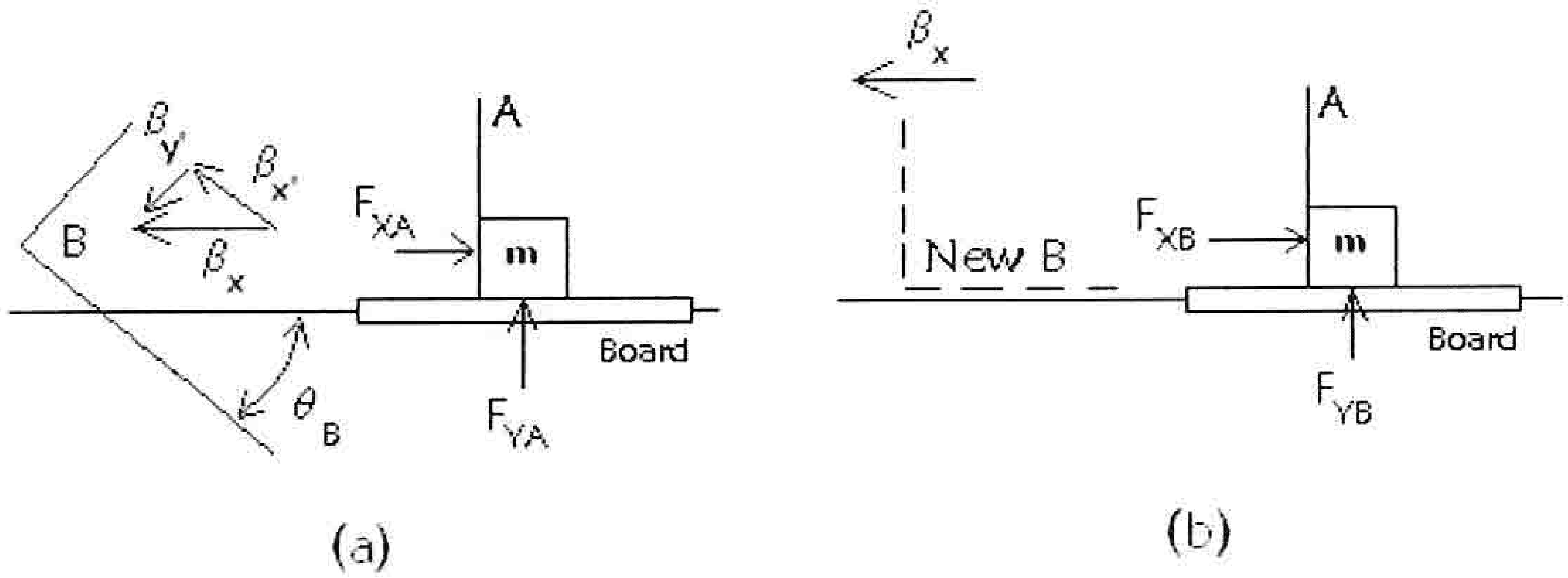


Figure 14. The experiment of Figure 13 as seen by frame A.

Equation (46) applies to any experiment where forces that a mass "feels" are known and the forces from an applying reference frame are desired.

Conclusions

The Special Relativity transformations for forces applied between inertial reference frames are determined by Newton's Laws. Reference frame choice is important in force transformation due to the possible observation of "mysterious forces" in the observing reference frame. Mysterious forces can arise if the mass receiving the force is not stationary in the observing reference frame. Mysterious forces are related to the distortion in geometry (length contraction, failure of simultaneity at a distance) between inertial reference frames. Force and geometry are linked within Special Relativity.

This link establishes a fourth principle of Special Relativity - "Force Enhancement" given by (23). Because force enhancement is a fundamental operating principle of Special Relativity, it provides a simple way to transform forces operating between inertial reference frames ((45) and (46)). Forces differ from the quantities describe by the other three principles of Special Relativity in that forces are absolute (not relative) and are not directly observable except in the reference frames where they originate and terminate.